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One-Dimensional Space Charge Waves – Small Signal Theory and Applications

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This paper gives an introduction to one-dimensional plasma oscillations and space charge waves. As applications, the plasma resonance between two parallel plates is considered and the beam current modulation produced in a klystron by a modulating cavity resonator is determined.

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1. Introduction

In 1929, Lewi Tonks and Irwing Langmuir observed electron oscillations in ionized gases [1]. While the pure existence of oscillations in electron clouds has not been much surprising – oscillations in air have been well known this time – its properties have been: In first order, the frequency of these oscillations depend on the electron density but not on the boundary conditions. Thus, in contrast to oscillations in air, the oscillation frequency of a plasma is a property of the medium and not a property of the surrounding geometry.

These oscillations and other phenomenons appearing in electron clouds may be explained by the space charge wave theory, which applies MAXWELL's equations and the force equation of LORENTZ.

2. The eigenfrequency phenomenon

Assume an infinitely extended plasma with electron charge density ϱ_0 and ion charge density $-\varrho_0$. Due to the large inertia of the ions, if an rf field acts on the plasma, with respect to the electrons the motion of the ions can be neglected. If the electrons between $-z_0$ and z_0 are compressed at both sides to the area between $-z$ and z (figure 1), the electron charge remains constant while the charge of the stationary ions changes

$$\begin{aligned} Q_e &= \varrho_0 A 2 z_0 \\ Q_i &= -\varrho_0 A 2 z. \end{aligned}$$

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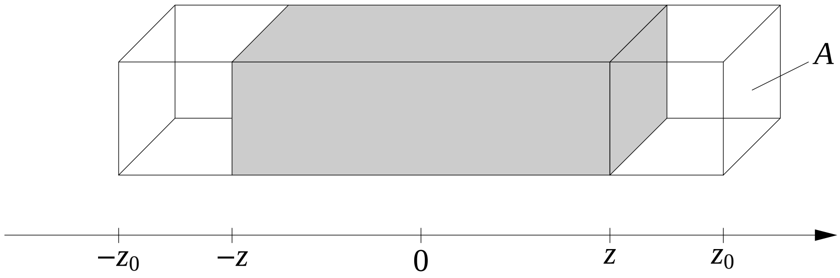

Figure 1: One-dimensional electron cloud excited by compression.

The total charge between $-z$ and z is therefore

$$Q(z) = 2A\varrho_0(z_0 - z).$$

Since there are no variations in x and y, there will be a z-component of the electric field only and Gauss law yields the simple relation

$$Q(z) = \varepsilon_0 A[E(z) - E(-z)].$$

From the symmetry of the problem, $E(-z) = -E(z)$ and thus

$$E(z) = \frac{\varrho_0}{\varepsilon_0}(z_0 - z).$$

The acceleration of the electrons at z consequently reads

$$\ddot{z} = -\eta E(z) = \eta\frac{\varrho_0}{\varepsilon_0}(z - z_0)$$

yielding the homogeneous oscillation equation

$$\ddot{z} - \eta\frac{\varrho_0}{\varepsilon_0}(z - z_0) = 0$$

with the eigenfrequency of oscillation

$$\omega_p = \sqrt{-\eta\frac{\varrho_0}{\varepsilon_0}}$$

which is called *plasma frequency*. Here, $\eta = 1.759 \times 10^{11}$As/kg is the specific electron charge and ϱ_0 the (negative) electron charge density. Typical values for the plasma frequency of electron beams used in klystrons are $\omega_p = 1 \dots 10$GHz.

3. Theory of one-dimensional space charge waves

Space charge waves are related rather to hydrodynamics than to electrodynamics. The fundamental laws for their derivation are

- MAXWELL's equations (divergence and continuity),
- LORENTZ's force equation,

- and the convection current equation.

Instead of starting with the fundamental hydrodynamic laws (see [2], chapter 10), since I am interested in wave solutions only, I will assume forward and backward waves for all components under consideration. Afterwards these fields will be checked against the laws mentioned above which will result in conditions for the wave amplitudes.

3.1. The field ansatz

The fields under consideration are the charge density, the velocity, the (convection) current density, the electric and the magnetic field, which are functions of z and t only; the vectorfields consist of a z-component only. All fields consist of a stationary (subscript 0), a forward (subscript F) and a backward (subscript B) wave part[1]. Furthermore, the magnetic field is assumed to be zero[2].

$$\begin{aligned}
\varrho(z,t) &= \varrho_0 + \varrho_F e^{j(\omega t - kz)} + \varrho_B e^{j(\omega t + kz)} && (1)\\
v(z,t) &= v_0 + v_F e^{j(\omega t - kz)} + v_B e^{j(\omega t + kz)} && (2)\\
J(v,z) &= J_0 + J_F e^{j(\omega t - kz)} + J_B e^{j(\omega t + kz)} && (3)\\
E(z,t) &= E_0 + E_F e^{j(\omega t - kz)} + E_B e^{j(\omega t + kz)} && (4)\\
H(z,t) &= 0 && (5)
\end{aligned}$$

These equations describe the electron motion only. Due to the large inertia of the ions, for the frequencies under consideration the corresponding fields for the ions are considered to consist of dc components only. Thus, ϱ_{ion}, v_{ion} and J_{ion} are constants and the total fields[3] may be written as

$$\begin{aligned}
\varrho_{\text{total}} &= \varrho(z,t) + \varrho_{ion}\\
J_{\text{total}} &= J(z,t) + J_{ion}.
\end{aligned}$$

3.2. Conditions for the wave amplitudes

3.2.1. The convection current equation

For the convection current density of the electrons

$$\mathbf{J} = \varrho \mathbf{v},$$

and with (1) and (2)

$$\begin{aligned}
J = {} & \varrho_0 v_0 + (\varrho_0 v_F + \varrho_F v_0) e^{j(\omega t - kz)} + (\varrho_0 v_B + \varrho_B v_0) e^{j(\omega t + kz)}\\
& + (\varrho_F v_F) e^{2j(\omega t - kz)} + (\varrho_B v_B) e^{2j(\omega t + kz)} + (\varrho_F v_B + \varrho_B v_F) e^{2j(\omega t)}.
\end{aligned}$$

Since these six components are lineary independent a comparison with equation (3) yields

$$\begin{aligned}
J_0 &= \varrho_0 v_0\\
J_F &= \varrho_0 v_F + \varrho_F v_0\\
J_B &= \varrho_0 v_B + \varrho_B v_0.
\end{aligned}$$

[1] For the theory it was enough to set up a forward wave only. However, in the application example section 4.1 we will need the backward wave to match our boundary conditions.

[2] This is not obvious, indeed. But what is the meaning of ansatz?

[3] A total velocity does not exist. Try to imagine what physical meaning such a field should have.

For the second order components we may write

$$
\begin{aligned}
J_{2F} &= \varrho_F v_F \\
J_{2B} &= \varrho_B v_B \\
J_{2S} &= \varrho_F v_B + \varrho_B v_F .
\end{aligned}
$$

To keep the analysis linear, we have to make a first small signal approximation. Usually, it is assumed that the magnitudes of the rf components are much smaller than these of the dc components. However, since this analysis shall be valid for stationary electron clouds also ($v_0 = 0$), we must be less restrictive here. We will assume

$$\varrho_F \ll \varrho_0 \quad \text{and} \quad \varrho_B \ll \varrho_0 \qquad \text{for} \quad v_0 = 0 \tag{6}$$

$$v_F \ll v_0 \quad \text{and} \quad v_B \ll v_0 \qquad \text{for} \quad v_0 \neq 0. \tag{7}$$

In this way it is guaranted that all second order rf components are much less in magnitude than the first order rf components which may therefore be neglected.

To be complete, the convection current density of the ions reads $\mathbf{J}_{ion} = \varrho_{ion}\mathbf{v}_{ion}$, which reduces to

$$J_{ion} = \varrho_{ion} v_{ion} .$$

3.2.2. The divergence equation

For the divergence equation we have

$$\varepsilon_0 \mathrm{div}\mathbf{E} = \varrho_{\mathrm{total}}$$

and with (4) and (1)

$$-\mathrm{j}k\varepsilon_0 (E_F \mathrm{e}^{\mathrm{j}(\omega t - kz)} - E_B \mathrm{e}^{\mathrm{j}(\omega t + kz)}) = \varrho_0 + \varrho_{ions} + \varrho_F \mathrm{e}^{\mathrm{j}(\omega t - kz)} + \varrho_B \mathrm{e}^{\mathrm{j}(\omega t + kz)}. \tag{8}$$

Since the dc component of the electric field may be chosen arbitrarily

$$
\begin{aligned}
E_0 &= 0 \\
E_F &= \mathrm{j}\frac{\varrho_F}{k\varepsilon_0} \\
E_B &= -\mathrm{j}\frac{\varrho_B}{k\varepsilon_0}.
\end{aligned}
$$

Additionally, equation (8) requires that

$$\varrho_0 = -\varrho_{ions}, \tag{9}$$

which is discussed in section 3.2.5.

3.2.3. The induction equation

The next law to fulfill is the induction equation which reads

$$\mathbf{curlH} = \mathbf{J}_{\mathrm{total}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{10}$$

Taking the divergence we have

$$\mathrm{j}k(J_F \mathrm{e}^{\mathrm{j}(\omega t - kz)} - J_B \mathrm{e}^{\mathrm{j}(\omega t + kz)}) = \omega k \varepsilon_0 (E_F \mathrm{e}^{\mathrm{j}(\omega t - kz)} - E_B \mathrm{e}^{\mathrm{j}(\omega t + kz)}),$$

$$\begin{aligned} J_F &= -\mathrm{j}\omega\varepsilon_0 E_F \\ J_B &= -\mathrm{j}\omega\varepsilon_0 E_B. \end{aligned}$$

Inserting these fields into (10) yields the surprising relation

$$\mathbf{curlH} = (J_0 + J_{ion})\mathbf{e}_z, \tag{11}$$

meaning that the displacement current totally compensates the rf convection current. Thus, no rf component of magnetic field is required. However, having the magnetic field vanishing totally, (11) requires that

$$J_0 = -J_{ion}, \tag{12}$$

which is discussed in section 3.2.5.

3.2.4. The force equation

Finally, Lorentz's force equation

$$\dot{\mathbf{p}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

must be satisfied. Since there is no magnetic field and motion is in z-direction only, it reduces to (see appendix A)

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{q}{m_0 \gamma^3} E.$$

With respect to the electrons, the specific charge q/m_0 of the ions may be neglected; the ion velocity is considered to be constant. However, the rest specific charge $-\eta_0 = -e/m_{e0}$ of the electrons is large and an rf electric field will considerably influence their speed.

With the relativistic specific electron charge $-\eta = -e/m_e$, inserting (2) and (4) into

$$\dot{v} = -\eta E/\gamma^2$$

yields

$$\mathrm{j}v_F \mathrm{e}^{\mathrm{j}(\omega t - kz)}(\omega - kv_0) + \mathrm{j}v_B \mathrm{e}^{\mathrm{j}(\omega t + kz)}(\omega + kv_0) = -\eta/\gamma^2 (E_F \mathrm{e}^{\mathrm{j}(\omega t - kz)} + E_B \mathrm{e}^{\mathrm{j}(\omega t + kz)}),$$

$$\begin{aligned} E_F &= -\mathrm{j}\frac{\omega - kv_0}{\eta/\gamma^2} v_F \\ E_B &= -\mathrm{j}\frac{\omega + kv_0}{\eta/\gamma^2} v_B. \end{aligned}$$

Here the total time derivative of the phase $\varphi = \omega t \pm kz$,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial\varphi}{\partial t} \pm \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial t} = \omega \pm kv(z,t)$$

and the small signal approximation

$$\omega \pm kv(z,t) \quad \approx \quad \omega \pm kv_0$$

which may be rewritten as

$$\begin{aligned} v_F \ll \omega/k \quad \text{and} \quad v_B \ll \omega/k \qquad \text{for} \quad v_0 = 0 \\ v_F \ll v_0 \quad \text{and} \quad v_B \ll v_0 \qquad \text{for} \quad v_0 \neq 0 \end{aligned} \tag{13}$$

have been applied.

3.2.5. Discussion of the ion contribution

Including space charge into unbounded considerations may lead to an inconsistency regarding the dc components of the electric and the magnetic field.

Starting with the electric field, let's homogeneously fill the universe with (non-vanishing) space charge. Having chosen a fixed coordinate system, the divergence equation requires an electric field of the type $E_r \propto r$. Since the charge distribution is the same in a system having its origin at a different point than the first, the field type must be the same. Thus, the field at a certain point in the universe depends on the system chosen, because the correspondig value of r is different for all systems with different origins.

Continuing with the magnetic field, let's additionaly assume the charge to drift into z-direction. Since this means a non-vanishing current density with respect to the fixed coordinate system, the induction law requires a magnetic field of the type $H_\varphi \propto \varrho$. Again, the field at a certain point depends on the system chosen, since the current distribution is the same for all systems having parallel z-axis'.

Now, physics requires solutions beeing independent of the coordinate system. Together with Maxwell's equations we must conclude that the concept of a space charge filled universe is inacceptable. However, reducing analysis to one dimension (and thus accepting unbounded areas) means a significant simplification. Furthermore, in first order we are interested in the rf components of the fields only, for which the inconsistency does not occur. And since the coordinate system independency appears to be more stringend than Maxwell's equations we will simply assume vanishing dc electric and magnetic fields, accepting the inconsistency regarding the divergence and induction laws.

For our special case, since we are dealing with a plasma, the average charge density vanishes and equation (9) holds anyway. However, instead of meeting (12) we set $v_{ions} = 0$ which yields in $\mathbf{J}_{ions} = \mathbf{0}$ and the inconsistency for the static magnetic field remains.

3.3. The general solution

In result of applying all relevant laws to the field ansatz (1) to (5) we have derived two homogeneous systems of four linear equations for the four field qualities, charge density, velocity, current density and electric field, one for the forward and one for the backward wave

$$\begin{pmatrix} v_0 & \varrho_0 & -1 & 0 \\ 1 & 0 & 0 & \pm \mathrm{j}k\varepsilon_0 \\ 0 & 0 & 1 & \mathrm{j}\omega\varepsilon_0 \\ 0 & \mathrm{j}\gamma^2(\omega \mp kv_0)/\eta & 0 & 1 \end{pmatrix} \begin{pmatrix} \varrho_{F/B} \\ v_{F/B} \\ J_{F/B} \\ E_{F/B} \end{pmatrix} \quad = \quad \mathbf{0}, \tag{14}$$

with the corresponding determinants

$$\Delta_{F/B} = -\frac{\varepsilon_0\gamma^2}{\eta}\left((\omega \mp kv_0)^2 + \frac{\varrho_0\eta}{\varepsilon_0\gamma^2}\right).$$

For having non-trivial solutions the determinants must vanish

$$(\omega \mp kv_0)^2 = \omega_p^2, \tag{15}$$

where the more general definition of the *plasma frequency* has been used

$$\omega_p = \sqrt{-\frac{\varrho_0\eta}{\varepsilon_0\gamma^2}}.$$

Obviously, not all of the constants ϱ_0, v_0, ω and k may be chosen arbitrarily. From a practical point of view we assume the dc charge density and the dc velocity to be given. Depending on the value of v_0, equation (15) determines the frequency or the wave number.

Equations (14) and (15) represent the most general solution of the one-dimensional plasma oscillation problem. For a set of values ϱ_0, v_0, ω and k meeting (15), the systems (14) give the relations between the wave amplitudes of the fields. Their absolute values are finally determined by the boundary conditions of the problem, which are usually given in terms of the velocity. Therefore, we will express all (other) wave amplitudes as multiples of the velocity wave amplitudes. This procedure is splitted into two parts, since the relations for a drifting cloud significantly differ from those valid for stationary clouds.

3.3.1. Stationary electron clouds

For a stationary cloud we have $v_0 = 0$. Except for the charge density all dc fields vanish and equation (15) reads

$$\omega = \omega_p.$$

I.e., the only frequency at which a (not externally driven) stationary electron cloud may oscillate is *its plasma frequency* no matter what boundary conditions apply.

Thus, the general solution for a stationary electron cloud is given by equations (1) to (5) with $\omega = \omega_p$ and

$$\begin{pmatrix} v_0 & \varrho_0 & -1 & 0 \\ 1 & 0 & 0 & \pm \mathrm{j}k\varepsilon_0 \\ 0 & 0 & 1 & \mathrm{j}\omega_p\varepsilon_0 \\ 0 & \mathrm{j}\omega_p/\eta & 0 & 1 \end{pmatrix}\begin{pmatrix} \varrho_{F/B} \\ v_{F/B} \\ J_{F/B} \\ E_{F/B} \end{pmatrix} = \mathbf{0} \tag{16}$$

or

$$\begin{aligned} \varrho_{F/B} &= \pm\varrho_0\frac{k}{\omega_p}v_{F/B} \\ J_{F/B} &= \varrho_0 v_{F/B} \\ E_{F/B} &= \mathrm{j}\varrho_0\frac{1}{\varepsilon_0\omega_p}v_{F/B}. \end{aligned} \tag{17}$$

With (17), the small signal approximation (6) reduces to (13), namely

$$v_{F/B} \ll \frac{\omega_p}{k}, \tag{18}$$

which is the final condition to fulfill.

3.3.2. Drifting electron clouds (beams)

The situation for drifting clouds is a little bit more complicated than for stationary clouds, since for $v_0 \neq 0$ equation (15) has four solutions

$$k = \pm k_{s/f}$$

with

$$k_{s/f} = \frac{\omega \pm \omega_p}{v_0}.$$

The indices s and f refer to a *slow* and a *fast space charge wave* (phase velocitiy), respectively. Obviously, two more linear independent waves appear in the analysis and we could rewrite the field ansatz covering all four waves: the slow forward, the fast forward, the slow backward and the fast backward wave. However, for the application example section 4.2 the backward waves are not needed, so we modifiy the field ansatz to

$$\begin{aligned}
\varrho(z,t) &= \varrho_0 + \varrho_s e^{j(\omega t - k_s z)} + \varrho_f e^{j(\omega t - k_f z)} && (19)\\
v(z,t) &= v_0 + v_s e^{j(\omega t - k_s z)} + v_f e^{j(\omega t - k_f z)} && (20)\\
J(v,z) &= J_0 + J_s e^{j(\omega t - k_s z)} + J_f e^{j(\omega t - k_f z)} && (21)\\
E(z,t) &= E_0 + E_s e^{j(\omega t - k_s z)} + E_f e^{j(\omega t - k_f z)} &&\\
H(z,t) &= 0. && (22)
\end{aligned}$$

Taking all steps from section 3.2 again, we derive two linear systems for the waves amplitudes, one for the fast and one for the slow (forward) space charge wave:

$$\begin{pmatrix} v_0 & \varrho_0 & -1 & 0 \\ 1 & 0 & 0 & jk_{s/f}\varepsilon_0 \\ 0 & 0 & 1 & j\omega\varepsilon_0 \\ 0 & j\gamma^2(\omega - k_{s/f}v_0)/\eta & 0 & 1 \end{pmatrix} \begin{pmatrix} \varrho_{s/f} \\ v_{s/f} \\ J_{s/f} \\ E_{s/f} \end{pmatrix} = \mathbf{0}. \qquad (23)$$

Thus, the general solution for a drifting electron cloud is given be equations (19) to (22) with the amplitudes related by (23) or

$$\begin{aligned}
\varrho_{s/f} &= \mp \frac{\varrho_0}{v_0} \frac{\omega \pm \omega_p}{\omega_p} v_{s/f} && (24)\\
J_{s/f} &= \mp \varrho_0 \frac{\omega}{\omega_p} v_{s/f} &&\\
E_{s/f} &= \mp j \varrho_0 \frac{1}{\varepsilon_0 \omega_p} v_{s/f}. && (25)
\end{aligned}$$

All equations are valid as long as the small signal approximation (7) holds, that is

$$v_{s/f} \ll v_0. \qquad (26)$$

4. Application of one-dimensional space charge waves

4.1. Resonators

Consider the space between two infinitely extended parallel plates, one at $z = -a$ and the other at $z = a$, to be filled with non-drifting plasma. What (non-trivial) waves may exist within this arragement?

Since the electrons cannot move through the plates we have the simple boundary condition

$$v(-a,t) \quad = \quad v(a,t) \quad = \quad 0$$

and (2) yields the linear system

$$\begin{pmatrix} e^{jka} & e^{-jka} \\ e^{-jka} & e^{jka} \end{pmatrix} \begin{pmatrix} v_F \\ v_B \end{pmatrix} \quad = \quad \mathbf{0}.$$

For having non-trivial solutions the determinant $2j \sin 2ka$ must vanish, i.e.

$$k = \frac{m\pi}{2a}, \qquad m = 1, 2, \ldots$$

and the system yields

$$v_B \quad = \quad v_F(-1)^{m+1}$$

having for the fields for odd values of m

$$\begin{aligned} \varrho(z,t) &= \varrho_0 - 2jv_F \frac{\varrho_0}{\omega_p} \frac{m\pi}{2a} \sin \frac{m\pi z}{2a} e^{j\omega_p t} \\ v(z,t) &= 2v_F \cos \frac{m\pi z}{2a} e^{j\omega_p t} \\ J(z,t) &= 2v_F \varrho_0 \cos \frac{m\pi z}{2a} e^{j\omega_p t} \\ E(z,t) &= 2jv_F \frac{\varrho_0}{\varepsilon_0 \omega_p} \cos \frac{m\pi z}{2a} e^{j\omega_p t} \\ H(z,t) &= 0. \end{aligned}$$

For even values of m, the terms cos and $-j \sin$ must be exchanged. These fields fulfill MAXWELL's equation as long as the small signal approximation (18) holds, i.e.

$$v_F \quad \ll \quad \omega_p \frac{2a}{m\pi}$$

4.2. Bunching of a velocity modulated electron beam

Consider an (infinitely extended) electron beam of the homogeneous charge density ϱ_0 drifting along the z-axis with velocity v_0, figure 2. At $z = 0$ we have a modulator causing the velocity to jump from v_0 to $v_0 + \hat{v} e^{j\omega t}$. How do the charge and curent densities behave for $z > 0$?

There are two boundary conditions to apply to the general solution from section 3.3.2: the jump in rf velocity at $z = 0$ and that there is no rf charge density at $z = 0$[4]. Starting with the second condition

$$\varrho(0,t) \quad = \quad \varrho_0,$$

equation (19) immediately yields

$$\varrho_s + \varrho_f \quad = \quad 0$$

[4]Sometimes this condition is replaced by a vanishing rf current density which yield slightly different equations but the same final result.

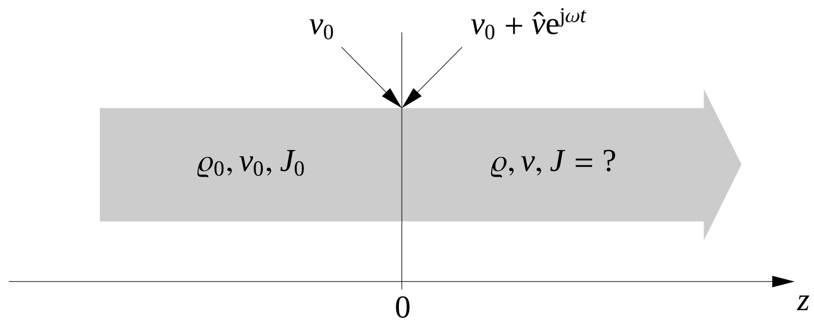

Figure 2: One dimensional electron beam with velocity modulator in $z = 0$.

which – together with (24) – means for the velocity wave amplitudes

$$(\omega + \omega_p)v_s - (\omega - \omega_p)v_f \quad = \quad 0. \tag{27}$$

Continuing with the jump condition

$$v(0, t) \quad = \quad v_0 + \hat{v}e^{j\omega t},$$

equation (20) yields

$$v_s + v_f \quad = \quad \hat{v}. \tag{28}$$

The solution of the linear system (27), (28) determines the velocity wave amplitudes so that with (24) to (25) all wave amplitudes are given

$$\begin{aligned}
\varrho_{s/f} &= \mp\varrho_0 \frac{(\omega + \omega_p)(\omega - \omega_p)}{\omega\omega_p} \frac{\hat{v}}{2v_0} \\
v_{s/f} &= \frac{\omega \mp \omega_p}{\omega} \frac{\hat{v}}{2} \\
J_{s/f} &= \mp\varrho_0 \frac{\omega \mp \omega_p}{\omega_p} \frac{\hat{v}}{2} \\
E_{s/f} &= \mp j\varrho_0 \frac{1}{\varepsilon_0\omega_p} \frac{\omega \mp \omega_p}{\omega} \frac{\hat{v}}{2}.
\end{aligned} \tag{29}$$

With the *plasma wavenumber* and the usual wavenumber

$$\begin{aligned}
k_p &= \omega_p / v_0 \\
k &= \omega / v_0
\end{aligned}$$

the interesting rf fields follow from eqns. (19) to (21)

$$\begin{aligned}
\tilde{\varrho}(z, t) &= j\varrho_0 \frac{\hat{v}}{v_0} \frac{(\omega + \omega_p)(\omega - \omega_p)}{\omega\omega_p} \sin(k_p z) e^{j(\omega t - kz)} \\
\tilde{v}(z, t) &= \hat{v}\Big(\cos(k_p z) + j\frac{\omega_p}{\omega} \sin(k_p z)\Big) e^{j(\omega t - kz)} \\
\tilde{J}(z, t) &= j\varrho_o \hat{v} \frac{\omega}{\omega_p}\Big(\sin(k_p z) - j\frac{\omega_p}{\omega} \cos(k_p z)\Big) e^{j(\omega t - kz)}.
\end{aligned}$$

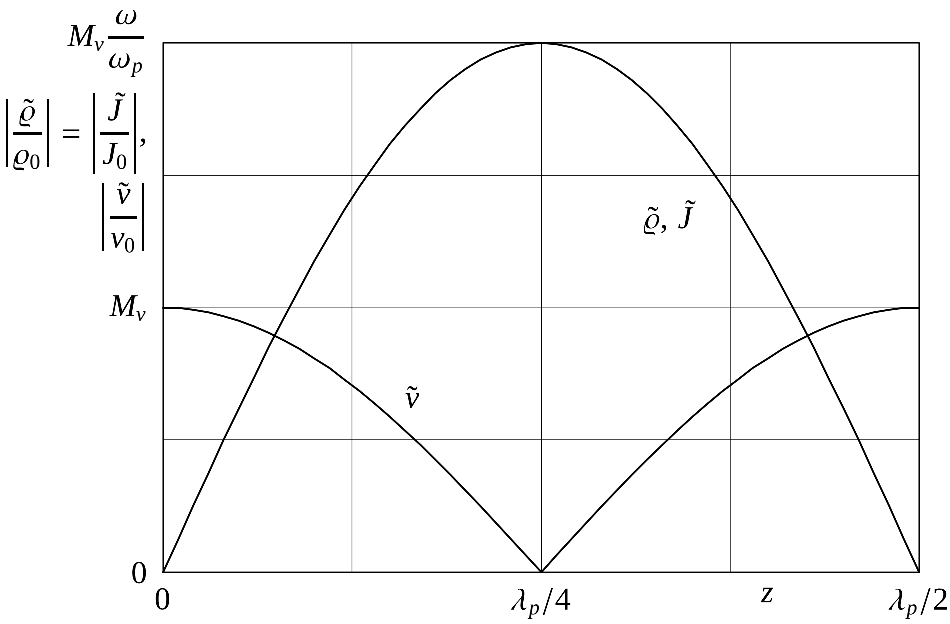

Figure 3: Normalized magnitudes of the rf fields in a one-dimensional velocity modulated electron beam.

A usual assumption for electron beam mircowave devices is that the plasma frequency is much less than the operating frequency

$$\omega_p \ll \omega \tag{30}$$

and the rf fields become even more simple

$$\tilde{\varrho}(z,t) \approx \mathrm{j}\varrho_0 \frac{\hat{v}}{v_0}\frac{\omega}{\omega_p}\sin(k_p z)\mathrm{e}^{\mathrm{j}(\omega t - kz)} \tag{31}$$

$$\tilde{v}(z,t) \approx \hat{v}\cos(k_p z)\mathrm{e}^{\mathrm{j}(\omega t - kz)}$$

$$\tilde{J}(z,t) \approx \mathrm{j}\varrho_o \hat{v}\frac{\omega}{\omega_p}\sin(k_p z)\mathrm{e}^{\mathrm{j}(\omega t - kz)}. \tag{32}$$

By means of (29) and (30), the small signal condition (26) may be rewritten as

$$M_v = \frac{\hat{v}}{v_0} \ll 2,$$

saying that the *velocity modulation* impressed at $z = 0$ must be small enough. Figure 3 depicts the magnitudes of the normalized rf fields. Starting at $z = 0$, while the rf velocity decreases with increasing z, the rf charge and current densities increase. With the *plasma wavelength* defined as

$$\lambda_p = 2\pi v_0/\omega_p,$$

the densities reach their maximum at $z = \lambda_p/4$, where the rf velocity vanishes. Defining a current density amplitude as

$$\hat{J} = \varrho_0 \hat{v}\frac{\omega}{\omega_p}\sin(k_p z)$$

the *current modulation* is given by

$$M_I = \frac{\hat{J}}{J_0} = M_v \frac{\omega}{\omega_p} \sin(k_p z).$$

Another interesting relation can be derived by comparing of (31) with (32):

$$\tilde{J}(z,t) \approx v_0 \tilde{\varrho}(z,t).$$

In first order, the rf current sets up from the dc beam velocity and the rf charge density; the contribution of the rf velocity field to the rf beam current is neglectible small. And this is the case not only near $z = \lambda_p/4$ where the rf velocity vanishes but all over the beam. Obviously, this results from the small signal condition (26).

Finally, let's discuss the current modulation value. For a 25kV 10mm×0.3mm sheet electron beam carrying 1A we have a plasma frequency $\omega_p \approx$ 8GHz. With an operating frequency of f = 94GHz the frequency ratio becomes $\omega/\omega_p \approx 74$. To achieve a current modulation of $M_I = 1$ at $z = \lambda_p/4$ we need a velocity modulation of $M_v \approx 0.01$ which certainly fulfills the small signal condition. However, even $M_v = 0.02$ would fulfill that condition yielding a current modulation of approximately 2. But this meant that at certain instances of time the total current would become negative, since the rf part is larger than the dc current, which is physically impossible. Thus, we have to conclude that – even if Maxwell's equations and the small signal condition are satisfied whith such a current function – for having solutions with physical meaning, we must restrict the current modulation be less or equal to 1

$$M_v \leq \omega_p/\omega.$$

This also means to keep the charge density non-negative.

From another point of view, it is known that the maximum modulation of a non-negative current is 2. Why do we have to restrict our modulation to 1? – This results from the small signal condition (7). A current modulation > 1 is possible only with higher harmonics, for $M_I = 2$ all higher harmonic modulation coefficients are 2 also. But we restricted the analysis to keep the higher harmonics neglectible small; our small signal analysis is not capable of handling such a case.

A. The relativistic force equation

In section 3.2.4 we made use of the relation

$$\dot{p} = m_0 \gamma^3 \dot{v},$$

the correctness of which is shown in the following. We will need the following laws:

$$\begin{aligned} \mathbf{p} &= m\mathbf{v} \\ m &= m_0 \gamma \\ \gamma &= (1-\beta^2)^{-1/2} \\ \beta &= v/c_0. \end{aligned}$$

Here we go:

$$\dot{\mathbf{p}} = m_0 \dot{\gamma} \mathbf{v} + m_0 \gamma \dot{\mathbf{v}}. \quad (33)$$

The only complex term is $\dot{\gamma}$:

$$\begin{aligned}\dot{\gamma} &= \frac{\mathrm{d}}{\mathrm{d}t}(1-\beta^2)^{-1/2} \\ &= -\frac{1}{2}(1-\beta^2)^{-3/2}(-2\beta)\dot{\beta} \\ &= \gamma^3\beta\dot{\beta}.\end{aligned}$$

Since $\dot{\beta}v = \beta\dot{v}$, it follows for the magnitudes of equation (33)

$$\begin{aligned}\dot{p} &= m_0\gamma^3\beta^2\dot{v} + m_0\gamma\dot{v} \\ &= m_0\gamma^3(1-1/\gamma^2)\dot{v} + m_0\gamma\dot{v},\end{aligned}$$

$$\dot{p} = m_0\gamma^3\dot{v},$$

which is just the equation that was to derive.

In general, this equation is valid in its scalar appearance only. From equation (33) it immediately follows that the directions of $\dot{\mathbf{p}}$ and $\dot{\mathbf{v}}$ are equal if and only if $\dot{\gamma} = 0$, i.e. for constatnt particle energy. Thus

$$\dot{\mathbf{p}} \neq m_0\gamma^3\dot{\mathbf{v}},$$

References

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