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Spreading and Perveance of Round and Sheet Electron Beams

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Abstract

In the design of tubes applying long electron beams, a first step is to check the beam against its spreading capability. This paper discusses the spreading of infinitely long round and sheet beams moving in free space where no external fields are present. Formulas are derived for the beam thickness near the waist. It turns out that the spread is approximately proportional to beam perveance and the square of longitudinal distance.

1 INTRODUCTION

Even in the absence of external fields, a charged particle beam tends to spread due to electrostatic repulsion forces between the particles which generally exceed the magnetic attraction forces.

To keep the task of deriving formulas for the beam shape straight forward to solve, some general assumtions are made:

1) The beam moves in free space, no external fields are present.

2) The beam is (infinitely) long and its diameter alters only slightly with distance.

3) There is a plane of convergence.

4) The charge density within the beam at the plane of convergence is constant.

5) Considerations are restricted to the neighborhood of the plane of convergence.

The plane of convergence is a plane in which all electrons move with the same speed in the same direction. Assuming the existence of such a plane together with assumtion 2) ensures at least three important beam properties: First, trajectories do not cross. Second, together with assumtion 4), the charge density will depend on the downstream position only. And third, again with 4), beams which are round or sheet at the plane of convergence are round and sheet, respectively, at each plane parallel to the plane of convergence. Hence, the beam shape is given by the trajectories of the surface electrons.

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2 ROUND BEAMS

First we have to determine the fields at the beam surface. Consider the charge contained between the plane of convergence z_0 and an arbitrary chosen plane z, figure 1. Since the charge density ϱ depends on z only we have

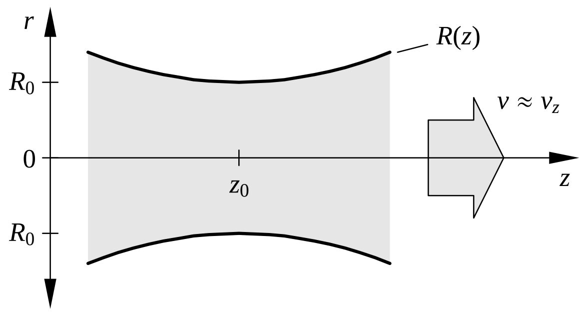

Figure 1: Section of a round beam.

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = \int_0^{2\pi}\int_0^{R} \varrho r \mathrm{d}r \mathrm{d}\varphi = \pi\varrho R^2.$$

From assumtion 2), near z_0 there will be no z-component of the electric field and since ϱ doesn't depend on φ, the electric field will finally have a radial component only beeing independent of φ. Then, the divergence theorem yields

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = \varepsilon_0 \int_0^{2\pi} E_r R \mathrm{d}\varphi = 2\pi\varepsilon_0 E_r R.$$

Hence

$$\mathbf{E} = \frac{\varrho R}{2\varepsilon_0}\mathbf{e}_r. \tag{1}$$

For the same reasons, the magnetic field has an azimuthal component only

$$\mathbf{B} = \frac{\mu_0 I}{2\pi R}\mathbf{e}_\varphi, \tag{2}$$

where $I = \pi R^2 \varrho v_z$ is the current through a plane $z = const.$ which is counted positive in z-direction. Replacing the charge density in equation (1) to be conform with (2)

$$\mathbf{E} = \frac{I}{2\pi\varepsilon_0 v_z R}\mathbf{e}_r.$$

Now that we have an approximation for the surface fields in the neighborhood of z_0, for setting up the equation for R we consider the force experienced by a surface electron. Taking $v \approx v_z$ into account this reads $\mathbf{F} = -e(E_r - vB_\varphi)\mathbf{e}_r$ and finally

$$\mathbf{F} = \frac{-eI}{2\pi\varepsilon_0 vR}\left(1 - \frac{v^2}{c_0^2}\right)\mathbf{e}_r.$$

Remembering I as a negative quantity, some important facts can be learned already from the above equations: First, the only velocity component that alters is v_r. Second, the electric force is defocusing while the magnetic force focuses. Since $v < c_0$, the net force is always defocusing. However, for relativistic beams it becomes neglectible and no beam spreading will occur. And finally third, the defocusing force decreases as the beam diameter increases.

With $F_r = m_0\gamma\ddot{R}$ from relativistic mechanics we have an initial value problem for $R(t)$ which can be immediately transformed to an initial value problem for $R(z)$ by means of $z = z_0 + v(t - t_0)$ yielding[1]

$$\begin{aligned} R''R &= \frac{-\eta_0 I}{2\pi\varepsilon_0 v^3\gamma^3} && (3)\\ R(z_0) &= R_0 && (4)\\ R'(z_0) &= 0. && (5) \end{aligned}$$

Unfortunately, this problem cannot be solved analytically. However, an approximative solution valid in the neighborhood of z_0 can be derived by integrating equation (3) twice after setting $R = R_0$ (see appendix for details)

$$\frac{R - R_0}{R_0} = \frac{-\eta_0 I}{4\pi\varepsilon_0 v^3\gamma^3}\left(\frac{z - z_0}{R_0}\right)^2.$$

Obviously, this solution overestimates the spread since R'' was considered to be a constant.

A handy formula can be derived by introducing the beam perveance $K = -I/V^{3/2}$ making use of the identity $v^3\gamma^3 = (\gamma+1)^{3/2}(\eta_0 V)^{3/2}$ and putting the remaining constants into a new constant K_0:

$$\frac{\Delta R}{R_0} = \frac{K}{K_0}\left(\frac{2}{\gamma+1}\right)^{3/2}\left(\frac{\Delta z}{R_0}\right)^2. \qquad (6)$$

The meaning of $K_0 = 8\pi\varepsilon_0\sqrt{2\eta_0} \approx 132\mu\text{P}$ becomes clear by setting $\Delta R = \Delta z = R_0$: K_0 is the perveance of a low energy beam ($\gamma \approx 1$) which diameter doubles one length of a radius away from its plane of convergence.

3 SHEET BEAMS

For having a two dimensional problem again, we assume a beam of infinite width in y-direction, fig. 2. With the same assumtions as for round beams we have the following fields at the upper beam surface:

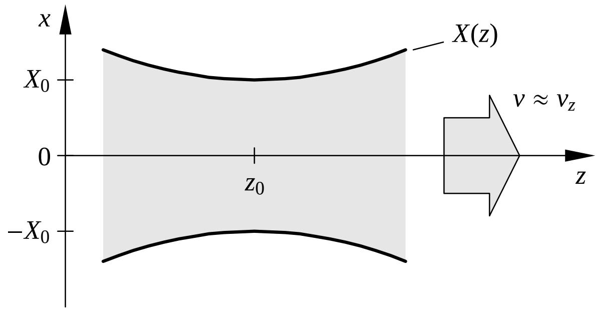

Figure 2: Section of a sheet beam.

$$\begin{aligned} \mathbf{E} &= \frac{I'}{2\varepsilon_0 v_z}\mathbf{e}_x\\ \mathbf{B} &= \frac{\mu_0 I'}{2}\mathbf{e}_y \end{aligned}$$

where $I' = 2X\varrho v_z$ is the current per width through a plane $z = const.$ counted positive in z-direction. Again, with $v \approx v_z$, the force experienced by an electron at the upper surface is

$$\mathbf{F} = \frac{-eI'}{2\varepsilon_0 v}\left(1 - \frac{v^2}{c_0^2}\right)\mathbf{e}_x.$$

In contrast to a round beam where the force decreases with increasing beam diameter, for sheet beams it is constant. While this fact is said to make actual beam design more complicated, it simplifies the analysis since the equation of motion becomes linear. With $F_x = m_0\gamma\ddot{X}$ we have

$$\begin{aligned} X'' &= \frac{-\eta_0 I'}{2\varepsilon_0 v^3\gamma^3}\\ X(z_0) &= X_0\\ X'(z_0) &= 0 \end{aligned}$$

which yields without any further approximations

$$\frac{X - X_0}{X_0} = \frac{-\eta_0 I' X_0}{4\varepsilon_0 v^3\gamma^3}\left(\frac{z - z_0}{X_0}\right)^2. \qquad (7)$$

For deriving a handy formula similar to (6) we could define a perveance per width as $K' = -I'/V^{3/2}$ but this solves neither for comparison purposes between round and sheet beams nor for having X_0 disappeared in the middle part of (7). Instead, we define a current as $I_\square = 2I'X_0$ which is called *current per square* since it is just the current carried by a partial beam of the width $2X_0$ which cross section is a square at $z = z_0$. Now, with $K_\square = -I_\square/V^{3/2}$ – the *perveance per square*[2] – we have a perveance value of usual

[1] That R corresponds to surface electrons entered the analysis only when substituting the charge density with the beam current. With corresponding current values these equations describe the motion of all beam particles.

[2] The perveance per square suffers from not beeing dependent on electrical beam parameters only. From gun design, the perveance per width depends in first order from the cathode to anode spacing only while the perveance per square depends on the focusing capability of the gun, additionally. Hence, it is a sheet beam parameter with respect to its spreading behaviour rather than a universal one.

dimension and the final formula becomes

$$\frac{\Delta X}{X_0} = \frac{\pi}{2}\frac{K_\square}{K_0}\left(\frac{2}{\gamma+1}\right)^{3/2}\left(\frac{\Delta z}{X_0}\right)^2. \tag{8}$$

Here, the constant K_0 is the same as in (6). Therefore, the perveance per square of a low voltage sheet beam which doubles in thinkness half the thickness away from its plane of convergence is $2K_0/\pi \approx 84\mu$P.

4 SAMPLE VALUES

Round beam. In the SLAC XK-5 klystron the beam values are approximately [4]: $R_0 = 1$cm, $I = 300$A, $V = 270$kV and $K = 2.14\mu$P. Therefore, the relative spread on 1cm was 1.14% from numerical integration, 1.14% from equation (6) and 1.6% neglecting γ. In the same order, the relative spread on $\lambda_p/4 = 3.9$cm was 16.9%, 17.3% and 24.7%, and the diameter would double after 10.0cm, 9.4cm and 7.8cm, respectively.

Sheet beam I. For the low energy sheet beam klystron worked on at our institute we have [5]: $X_0 = 0.15$mm, $I_\square = 0.03$A, $V = 25$kV and $K_\square = 0.008\mu$P. The relative spread on 0.15mm was 0.0092% from equation (8) and 0.0095% neglecting γ. On $\lambda_p/4 = 16.5$mm the spread was 111% and 115%, while the beam thickness doubles after 15.6mm and 15.4mm, respectively.

Sheet beam II. The beam values of the sheet beam klystron worked on by DULY Research are [6]: $X_0 = 4$mm, $I_\square = 19$A, $V = 300$kV and $K_\square = 0.116\mu$P. Therefore, the relative spread on 4mm was 0.09% from equation (8) and 0.14% neglecting γ. The relative spread on $\lambda_p/4 = 7.3$cm was 31% and 46%, while the beam thickness doubles after 13cm and 11cm, respectively.

5 CONCLUSIONS

The spread of round and sheet electron beams is proportional to the beam current and approximately the perveance, too. The beam diameter depends on the downstream position in a quadratic manner with its minimum at the plane of convergence which is also a symmetry plane regarding the beam shape.

Formulas for the relative spread of round and sheet beams valid near the waist are given by (6) and (8), respectively. While (6) slightly overestimates the spread (8) should be more exact. From comparing these two equations, the spreading behaviour of a round beam with a perveance K equals that of a sheet beam with a perveance per square of $2K/\pi$.

Although originally the main parameter of a space charge limited diode the perveance turns out to be an important beam parameter, too. Describing the beam spreading capability, it actually tells how expensive focusing will be.

A THE EQUATION $\ddot{X}X = C$

Beeing given the initial value problem

$$\begin{aligned} \ddot{x}x &= c \quad (9)\\ \dot{x}(t_0) &= 0 \quad (10)\\ x(t_0) &= x_0 \quad (11)\end{aligned}$$

with both the real constants c and x_0 greater than 0. Integration of $\mathrm{d}(\dot{x})^2 = 2\ddot{x}\mathrm{d}x = 2c/x\mathrm{d}x$ involving the initial values yields

$$\dot{x} = \pm\sqrt{2c\ln x/x_0} \tag{12}$$

or

$$t - t_0 = \pm\frac{1}{\sqrt{2c}}\int_{x_0}^{x}\frac{\mathrm{d}x'}{\sqrt{\ln x'/x_0}}. \tag{13}$$

This integral cannot be solved analytically which can be seen by transforming it according to $x = x_0\mathrm{e}^{u^2}$

$$t - t_0 = \pm\frac{2x_0}{\sqrt{2c}}\int_0^{\sqrt{\ln x/x_0}}\mathrm{e}^{u^2}\mathrm{d}u. \tag{14}$$

However, with the approximation $\ln x \approx x - 1$ the integral in (13) can be solved leading to

$$\frac{x - x_0}{x_0} \approx \frac{c}{2}\left(\frac{t-t_0}{x_0}\right)^2, \tag{15}$$

one approximative solution of our initial value problem valid in the neighborhood of t_0. Another solution follows from (14) with $\mathrm{e}^{u^2} \approx 1$

$$\frac{x - x_0}{x_0} \approx \exp\left[\frac{c}{2}\left(\frac{t-t_0}{x_0}\right)^2\right] - 1. \tag{16}$$

For the correct solution $x(t)$, the quadratic approximation $q(t)$ from (15) and the exponential approximation $e(t)$ from (16) we have the relations

$$\begin{aligned} x_0 &< x(t) \quad (17)\\ x(t) &< q(t) \quad (18)\\ q(t) &< e(t) \quad (19)\end{aligned}$$

for $t \neq t_0$ and $x(t_0) = q(t_0) = e(t_0) = x_0$. Prove:

Relation 17: Since for $t > t_0$ the positive sign applies in (13) and therefore in (12) also, we have $\dot{x} > 0$ and integration of $\dot{x}$ from t_0 to t yields a value greater than zero. For $t < t_0$ the negative sign applies but the integral keeps beeing greater than zero.

Relation 18: Since $\ddot{q} = c/x_0$, from (9) and (17) we have $\ddot{q} - \ddot{x} > 0$ for $t \neq t_0$. Integration from t_0 to t yields a value greater or less than zero for $t > t_0$ and $t < t_0$, respectively. Integrating again, we have (18) for $t \neq t_0$.

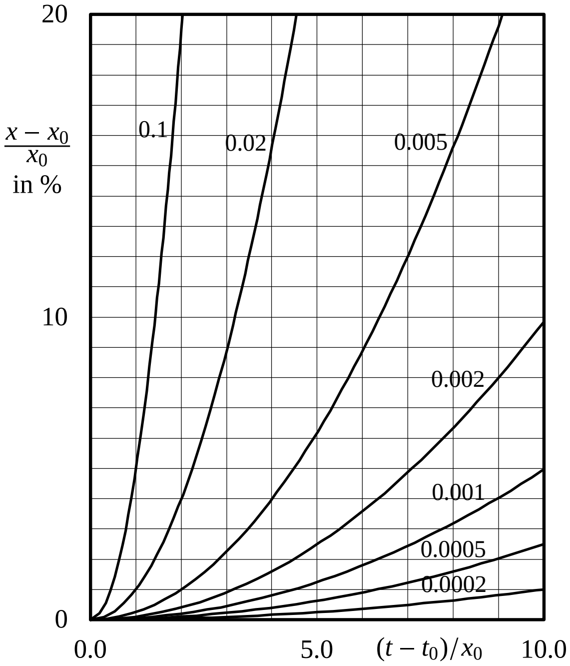

Figure 3: Correct solution of the initial value problem (9), (10), (11). Parameter is c.

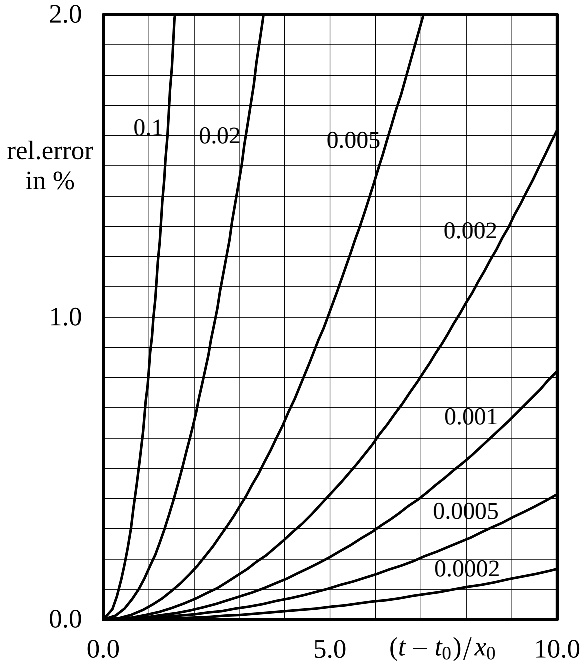

Figure 4: Relative error of quadratic approximation (15). Parameter is c.

Relation 19: With the well known TAYLOR series of $\exp(x)$ we have

$$e(t) - q(t) = \frac{y^2}{2!} + \frac{y^3}{3!} + \dots \tag{20}$$

where $y = c(t-t_0)^2/(2x_0^2)$. Obviously, (20) is positive for all $t \neq t_0$ and vanishes for $t = t_0$.

Thus, (15) is the better approximative solution of the initial value problem above and turned out to be a majorante of the correct solution.

In order to examine the suitability of (15), the correct solution was determined by solving (14) numerically. Figure 3 shows the result for some important values of c and figure 4 shows the relative error[3] of the quadratic approximation (15) for the same values c. With respect to beam spreading, from comparison of (9) with (3) the relation between c and the perveance is

$$c = \frac{K}{66\mu\mathrm{P}} \left(\frac{2}{\gamma+1}\right)^{3/2}.$$

For instance, the corresponding value for the XK-5 klystron from section 4 is $c = 0.023$.

Berlin, November 12 1997

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[3] The error is calculated for the normalized differences rather then the absolute values, i.e. for the relative spread rather than the diameter itself – when speaking in beam laguage. Thus, the error function is $(q-x)/(x-x_0)$.