PPM Focusing of Sheet Electron Beams

Steffen Solyga*
Technische Universität Berlin, EN-2, Einsteinufer 17, D-10587 Berlin, Germany

Abstract

This paper gives a simple two-dimensional analysis for the focusing of DC sheet electron beams by means of periodic permanent magnet (PPM) structures in three steps: 1. determination of the fields produced by a PPM assembly, 2. derivation of a general form of Busch's theorem, 3. determination of the beam envelopes. The formulas derived provide a possibility of fastly determining the induction needed for a given beam.

1 INTRODUCTION

In microwave tubes, beam focusing means compensating the radial space charge forces by means of auxilary fields. There are at least two advantages of using magnetic rather than electric fields: On the one hand, for fast electrons magnetic fields are capable of producing greater forces. For instance, a magnetic field of 1T which can be achieved with usual permanent magnets produces the same force on a $\beta = 0.3$ charge as an electric field of 100MV which was very expensive to generate. On the other hand, magnetic fields influence the electron pathes but do not modulate the beam. Thus, for focusing purposes magnetic fields are used.

The compressive force is usually achieved in two steps: Right behind the gun (sometimes even within the gun already) the beam is given a transverse velocity component by means of a transverse magnetic field. Then, a following longitudinal field provides the actual focusing.

The simplest and commonly applied focusing method for cylindrical beams – Brillouin focusing[1] – uses a solenoid enclosing the beam as shown in figure 1. At the entrance of this assembly the fringing field makes the beam rotate around its axis and the main field provides focusing. The induction needed to prevent the beam from scalloping is the Brillouin field $B_B = \sqrt{2}\omega_p/\eta$, where ω_p is the relativistic plasma frequency of the beam and η the relativistic charge to mass ratio for an electron. – This value will serve for reference later. – Unfortunately, Brillouin focusing cannot be used with sheet beams since the upper and lower electrons would drift in different directions.

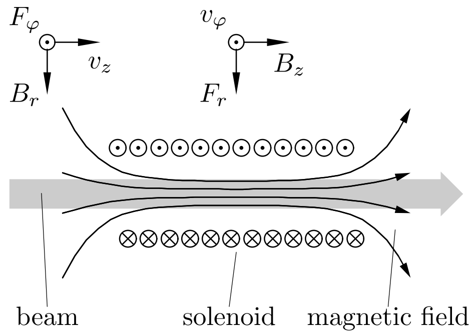

Figure 1: Brillouin focusing assembly (round beams)

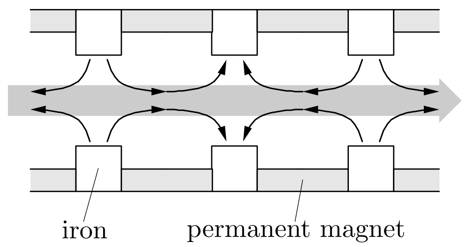

Figure 2: Periodic magnet focusing assembly

Another method – periodic magnet focusing – uses a couple of reversely poled magnets as shown in figure 2. One period may be thought of consisting of two solenoid sections with opposite fields. Since the transverse velocity component of each beam electron alters in sign from one half period to the next a net transverse drift does not appear. Thus, periodic focusing is well suited for sheet beams, too.

2 EVEN AND ODD STRUCTURES

Axial symmetry implies equal longitudinal fields across the beam surface at each single downstream position. In the planar case there are at least two possible types of symmetry since the lower half structure can be shifted with respect to the upper half, figures 3 and 4.

The first type corresponds to the axial symmetric case and is of the quality

$$B_x(-x,z) = -B_x(x,z) \tag{1}$$

$$B_z(-x,z) = B_z(x,z) \tag{2}$$

*solyga@tetibm2.ee.tu-berlin.de

[1] This name is due to Brillouin's important work on electron motion in magnetic fields, see [1].

Figure 3: Odd PPM assembly

Figure 4: Even PPM assembly

with $B_y = 0$. Since B_x is an odd function of x, I call this type an *odd* structure. Because of the field continuity, $B_x(0, z) = 0$.

The second type is of the quality

$$\begin{aligned} B_x(-x, z) &= B_x(x, z) && (3) \\ B_z(-x, z) &= -B_z(x, z) && (4) \end{aligned}$$

again with $B_y = 0$, which is referred to as an *even* structure. Here field continuity requires $B_z(0, z) = 0$.

From the fieldsymmetry follows a symmetric property of the trajectories. Assuming an electric field with an odd x-component and vanishing y-component, if $\mathbf{r} = (x, y, z)$ is a trajectory then $\mathbf{r}' = (-x, -y, z)$ and $\mathbf{r}' = (-x, y, z)$ are also trajectories for an odd and even structure, respectively, figures 5 and 6. Considering the symmetry of $y(x, z)$ with respect to x, we find that it is of the same type as that of the structure. Thus, an odd structure distorts a finite sheet beam with respect to y while an even structure preserves its shape.

3 FIELDS IN A PPM STRUCTURE

We simplify our considerations to an infinitively long (and wide) periodic structure with infinitively long permanent magnets. This should be quite realistic since a real structure will become much longer than the aperture and we are interested in the aperture fields only. Thus, one magnet can be represented by only

Figure 5: Trajectories in a magnetic field of odd symmetry

Figure 6: Trajectories in a magnetic field of even symmetry

one magnetic surface charge $\overline{\sigma} = \mu_0 M$ situated at the magnet pole, and the field by a harmonic scalar potential as $\mathbf{B} = -\mathbf{grad}\Phi$.

3.1 Odd structure

The reduced problem for an odd structure is shown in figure 7. While the conditions at $z = 0$ and $z = \lambda/2$ follow from the periodicity, the condition at $x = 0$ is just the property $B_x(0, z) = 0$ of an odd structure as already mentioned in section 2. Since the surface charge is an incontinuity we have to devide the expansion area into two subareas I and II. For both areas,

Figure 7: Boundary conditions for a quarter odd structure

the conditions at $z = 0$ and $z = \lambda/2$ can be fulfilled with a sine in z. The condition at $z = 0$ then requires a hyperbolic cosine in x for area I and the condition at $x \to \infty$ requires an exponential function in x for area II. Expanding the surface charge also we have

$$\begin{aligned} \Phi^I(x,z) &= \sum_{i=1}^{\infty} a_i \cosh(k_i x) \sin(k_i z) && (5) \\ \Phi^{II}(x,z) &= \sum_{i=1}^{\infty} b_i \exp(-k_i x) \sin(k_i z) && (6) \\ \sigma(z) &= \sum_{i=1}^{\infty} \sigma_i \sin(k_i z), && (7) \end{aligned}$$

where $k_i = 2\pi i/\lambda$ and

$$\sigma_i = \frac{4\overline{\sigma}}{\pi i} \sin(\pi i/2) \sin(\pi i d/\lambda). \quad (8)$$

The constants a_i and b_i can now be determined from the conditions at the charged surface $x = a/2$: Continuity of the potential requires

$$a_i \cosh(k_i a/2) = b_i \exp(-k_i a/2) \quad (9)$$

and the jump of its normal derivative requires

$$a_i k_i \sinh(k_i a/2) + b_i k_i \exp(-k_i a/2) = \sigma_i. \quad (10)$$

Thus we have

$$\begin{aligned} a_i &= \sigma_i/k_i \exp(-k_i a/2) && (11) \\ b_i &= \sigma_i/k_i \cosh(k_i a/2). && (12) \end{aligned}$$

In order to have the field expression as simple as possible, the higher order terms are checked against their importants. Recalling that $a_i = 0$ for even i and

$$a_i \propto 1/i^2 \sin(\pi i d/\lambda) \exp(-\pi i a/\lambda) \quad (13)$$

else, for a realistic structure one may have $\lambda = 4d = 10a$ yielding

$$\begin{aligned} |a_3/a_1| &= 6.05\% \\ |a_5/a_1| &= 0.56\% \\ |a_7/a_1| &= 0.05\%. \end{aligned}$$

Thus

$$\begin{aligned} \Phi(x,z) &\approx \Phi_1 \cosh\left(2\pi\frac{x}{\lambda}\right) \sin\left(2\pi\frac{z}{\lambda}\right) && (14) \\ \Phi_1 &= \frac{2\lambda\mu_0 M}{\pi^2} \sin\left(\pi\frac{d}{\lambda}\right) \exp\left(-\pi\frac{a}{\lambda}\right) && (15) \end{aligned}$$

is expected to be a quite good approximation of the true magnetic potential within the whole aperture[2].

[2]Although the expansion has been performed for $x > 0$ and $0 \le z \le \lambda/2$ only, the result is valid for the whole aperture $-a/2 \le x \le a/2$, $-\infty < z < \infty$, since the necessary information entered through the boundary condition already.

Figure 8: Boundary conditions for a quarter even structure

Finally, for the magnetic field we have

$$\begin{aligned} B_x &\approx -B_0 \sinh\left(2\pi\frac{x}{\lambda}\right) \sin\left(2\pi\frac{z}{\lambda}\right) && (16) \\ B_z &\approx -B_0 \cosh\left(2\pi\frac{x}{\lambda}\right) \cos\left(2\pi\frac{z}{\lambda}\right) && (17) \\ B_0 &= \frac{4\mu_0 M}{\pi} \sin\left(\pi\frac{d}{\lambda}\right) \exp\left(-\pi\frac{a}{\lambda}\right). && (18) \end{aligned}$$

3.2 Even structure

The even problem, as shown in figure 8, is attacked in the same manner as for the odd structure. The boundary conditions are matched by

$$\begin{aligned} \Phi^I(x,z) &= \sum_{i=1}^{\infty} a_i \sinh(k_i x) \sin(k_i z) && (19) \\ \Phi^{II}(x,z) &= \sum_{i=1}^{\infty} b_i \exp(-k_i x) \sin(k_i z) && (20) \end{aligned}$$

again with $k_i = 2\pi i/\lambda$. Continuity and the jump condition of the normal derivative at $x = a/2$ require

$$\begin{aligned} a_i \sinh(k_i a/2) = b_i \exp(-k_i a/2) && (21) \\ a_i k_i \cosh(k_i a/2) + b_i k_i \exp(-k_i a/2) = \sigma_i && (22) \end{aligned}$$

yielding

$$\begin{aligned} a_i &= \sigma_i/k_i \exp(-k_i a/2) && (23) \\ b_i &= \sigma_i/k_i \sinh(k_i a/2) && (24) \end{aligned}$$

where σ_i is still given by equation (8). Since (23) equals (11), within the aperture we have as good approximation

$$\begin{aligned} \Phi(x,z) &\approx \Phi_1 \sinh\left(2\pi\frac{x}{\lambda}\right) \sin\left(2\pi\frac{z}{\lambda}\right) && (25) \\ B_x &\approx -B_0 \cosh\left(2\pi\frac{x}{\lambda}\right) \sin\left(2\pi\frac{z}{\lambda}\right) && (26) \\ B_z &\approx -B_0 \sinh\left(2\pi\frac{x}{\lambda}\right) \cos\left(2\pi\frac{z}{\lambda}\right) && (27) \end{aligned}$$

where Φ_1 and B_0 are still given by (15) and (18), respectively.

4 A GENERAL FORM OF BUSCH'S THEOREM

The equations of motion for a free charge can be solved analytically only if certain restrictions apply to the fields. Even when assuming $\dot{z} = const.$ and $E_y = E_z = 0$ – this should be quite realistic for our focusing problem – these equations still read

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m_0 \dot{x}) &= -e(E_x + \dot{y}B_z - \dot{z}B_y) &(28)\\ \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m_0 \dot{y}) &= -e(\dot{z}B_x - \dot{x}B_z). &(29) \end{aligned}$$

It is not obvious that this problem can be solved at all.

Busch's theorem[3] is known of providing a solution for $\dot{\varphi}$ in the axial symmetric magnetic focusing problem. But this corresponds to the determination of $\dot{y}$ for our planar problem which is actually the basic problem for solving (28),(29). Therefore a general form of Busch's theorem in cartesian coordinates is derived in the following.

Problem: Solve the initial value problem

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m_0 \dot{\mathbf{r}}) &= -e(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}) &(30)\\ \mathbf{r}(t_0) &= \mathbf{r}_0 &(31)\\ \dot{\mathbf{r}}(t_0) &= \dot{\mathbf{r}}_0 &(32) \end{aligned}$$

for $\dot{y}(\mathbf{r})$ when the given fields are restricted to

$$\begin{aligned} E_y &= 0 &(33)\\ \partial\mathbf{B}/\partial t &= \mathbf{0} &(34)\\ \partial\mathbf{B}/\partial y &= \mathbf{0}. &(35) \end{aligned}$$

Solution: For the y component of equation (30) we have with (33)

$$\begin{aligned} \dot{p}_y &= -e\mathbf{e}_y \cdot (\dot{\mathbf{r}} \times \mathbf{B}), &(36)\\ p_y - p_{y0} &= -e\int_{t_0}^{t} (\mathbf{B} \times \mathbf{e}_y) \cdot \dot{\mathbf{r}}\mathrm{d}t. &(37) \end{aligned}$$

Since the integrand does not depend on the parameter t, equation (34), this is a parameter form of the line integral

$$p_y - p_{y0} = -e\int_{\mathcal{L}} (\mathbf{B} \times \mathbf{e}_y) \cdot \mathrm{d}\mathbf{r}, \qquad (38)$$

where $\mathcal{L}$ is the (unknown) uniquely defined trajectory solving the whole initial value problem. But this integral does not depend on the path of integration since $\mathbf{B} \times \mathbf{e}_y$ is a conservative field which follows with (35) from

$$\mathbf{curl}(\mathbf{B} \times \mathbf{e}_y) \equiv \partial\mathbf{B}/\partial y - \mathbf{e}_y \mathrm{div}\mathbf{B} \equiv \mathbf{0} \qquad (39)$$

[3] This name probably arises from his work on electronic motion in axial symmetric fields, see [2].

Figure 9: Electron trajectory and path of integration

and the legitimate assumtion that $\mathbf{B}$ is defined on a simply connected region. Thus we may choose a (well known) path making the integral easy to evaluate. Since normally the conditions at the cathode are known and the field is symmetric, the path of integration is usually chosen from the point of emission across the cathode to the plane of symmetry, along this plane and finally to the current electron position as shown in figure 9. Defining the relativistic charge to mass ratio as $\eta = e/\gamma m_0$, the problem is finally solved by

$$\dot{y} = \frac{\gamma_0}{\gamma}\dot{y}_0 - \eta\int_{\mathbf{r}_0}^{\mathbf{r}} (\mathbf{B} \times \mathbf{e}_y) \cdot \mathrm{d}\mathbf{r}, \qquad (40)$$

since its R.H.S. is a function of x and z only, indeed. This equation may be called a general representation of Busch's theorem.

Since the problem was solved in a mathematical manner, at least the integral from (38) should be given a physical meaning: Consider a vector $\mathbf{w} = w\mathbf{e}_y$ moving together with the charge along $\mathcal{L}$ thus defining a surface, figure 9. Since $\mathbf{w} \times \mathrm{d}\mathbf{r}$ is a differential element of this surface, recalling (35), the magnetic flux Ψ through this surface is

$$\Psi = \int_{\mathcal{L}} \mathbf{B} \cdot (\mathbf{w} \times \mathrm{d}\mathbf{r}) = w\int_{\mathcal{L}} (\mathbf{B} \times \mathbf{e}_y) \cdot \mathrm{d}\mathbf{r}. \qquad (41)$$

Defining $\Psi' = \Psi/w$, the physical meaning of the function

$$\Psi' = \int_{\mathcal{L}} (\mathbf{B} \times \mathbf{e}_y) \cdot \mathrm{d}\mathbf{r} \qquad (42)$$

turned out to be the magnetic flux per unit length through the surface generated by the trajectory together with $\mathbf{e}_y$.

For further information on Busch's theorem see [3] page 44 and [4] page 35.

5 CONDITIONS FOR SPACE CHARGE BLANCED FLOW

Consider a single beam electron – and its trajectory $\mathbf{r}(t)$ – which left the cathode at $t = t_0$ with $\dot{y}_0 =$

Figure 10: Trajectory, path of integration and considered fluxes

$\dot{y}(t_0) = 0$, has already passed the anode at $t = t_1$ with $\dot{x}_1 = \dot{x}(t_1) = 0$ and is now flying through the focusing structure. Since its speed results in first order from the DC acceleration we would expect that for $t \geq t_1$

$$v = |\dot{\mathbf{r}}| \approx \dot{z} \approx const. \tag{43}$$

Thus, its motion in z is simply

$$z = z_1 + v(t - t_1). \tag{44}$$

and for its motion in x we have

$$\ddot{x} = -\eta(E_x - vB_y + \dot{y}B_z). \tag{45}$$

Obviously, for sheet beams there are only two forces controlling the beam envelope: the defocusing force $E_x - vB_y$ which is due to the space charge and the focusing force $\dot{y}B_z$ arising from the field of the magnet assembly. If these forces are equal in magnitude but of opposite sign, the net force vanishes and the flow is said to be *space charge balanced*.

5.1 Uniform-Field focusing

As already mentioned in section 1, focusing with a uniform field cannot be applied with sheet beams. However, the analysis is straight forward and their results will serve for reference later.

As pointed out in some more detail in [5], the defocusing force is a constant and given by

$$E_x - vB_y = \frac{\varrho_1}{\varepsilon_0 \gamma^2} x_1, \tag{46}$$

where ϱ_1 is the charge density at the reference position $z_1 = z(t_1)$.

For determining the focusing force we now make use of Busch's theorem, equation (40), involving $\dot{y}_0 = 0$ and choosing the path of integration as shown in figure 10. Thus, Busch's theorem reads

$$\dot{y} = -\eta(\Psi_1' + \Psi_2' + \Psi_3' + \Psi_4') \tag{47}$$

where

$$\Psi_i' = \int\limits_{\mathcal{L}_i} (\mathbf{B} \times \mathbf{e}_y) \cdot \mathrm{d}\mathbf{r}. \tag{48}$$

If no flux links the cathode, Ψ_1' vanishes. But Ψ_2' and Ψ_3' also vanish since the fringing fields are symmetric to $x = 0$. And since the longitudinal field is simply $B_z = B_0$,

$$\Psi_4' = -xB_0 \tag{49}$$

$$\dot{y} = x\omega_0, \tag{50}$$

where $\omega_0 = \eta B_0$ has been introduced for convenience. Thus, the focusing force for uniform-field focusing reads

$$\dot{y}B_z = x\omega_0 B_0. \tag{51}$$

5.2 Periodic magnet focusing

If no assumtions are made on the the magnetic field in the gun except that no flux links the cathode, Ψ_1' vanishes but the constant Ψ_2' remains unknown. Ψ_3' and Ψ_4' can be determined from the known focusing fields by means of (48). Since for $x \ll 1$ $\sinh x \approx x$ and $\cosh x \approx 1$, assuming that

$$|x| \ll \lambda/2\pi, \tag{52}$$

which will be checked against its correctness later, the integration of (16), (17) and (26), (27) yields

$$\Psi_{3_{odd}}' = 0 \tag{53}$$

$$\Psi_{4_{odd}}' = B_0 x \cos\left(2\pi\frac{z}{\lambda}\right) \tag{54}$$

$$\Psi_{3_{even}}' = B_0 \frac{\lambda}{2\pi}\left[\cos\left(2\pi\frac{z}{\lambda}\right) - \cos\left(2\pi\frac{z_1}{\lambda}\right)\right] \tag{55}$$

$$\Psi_{4_{even}}' = 0. \tag{56}$$

Thus we have

$$-\dot{y}_{odd}/\eta = \Psi_{2_{odd}}' + B_0 x \cos\left(2\pi\frac{z}{\lambda}\right) \tag{57}$$

$$-\dot{y}_{even}/\eta = \Psi_{2_{even}}' - B_0\frac{\lambda}{2\pi}\cos\left(2\pi\frac{z_1}{\lambda}\right) + B_0\frac{\lambda}{2\pi}\cos\left(2\pi\frac{z}{\lambda}\right) \tag{58}$$

In order to prevent the beam from an average transverse drift the constants have to be zero. Thus the flux Ψ_2' must be chosen to

$$\Psi_{2_{odd}}' = 0 \tag{59}$$

$$\Psi_{2_{even}}' = B_0\frac{\lambda}{2\pi}\cos\left(2\pi\frac{z_1}{\lambda}\right) \tag{60}$$

and the equations finally read

$$\dot{y}_{odd} = -x\omega_0\cos\left(2\pi\frac{z}{\lambda}\right) \tag{61}$$

$$\dot{y}_{even} = -\frac{\lambda}{2\pi}\omega_0\cos\left(2\pi\frac{z}{\lambda}\right), \tag{62}$$

where $\omega_0 = \eta B_0$ has been introduced for convenience. Together with the longitudinal fields

$$B_{z_{odd}} = -B_0 \cos\left(2\pi \frac{z}{\lambda}\right) \quad (63)$$

$$B_{z_{even}} = -B_0 \frac{2\pi x}{\lambda} \cos\left(2\pi \frac{z}{\lambda}\right) \quad (64)$$

the focusing force turns out to be the same for both structure types

$$\dot{y} B_z = x \omega_0 B_0 \cos^2\left(2\pi \frac{z}{\lambda}\right). \quad (65)$$

However, recalling (52) the contributions of the velocities and the fields to this force are very different: While for an odd structure the transverse motion is very small and the longitudinal field is strong, for an even structure the motion is considerable and the longitudinal field is weak.

Finally, with the two forces derived (45) now reads

$$\ddot{x} + x\omega_0^2 \cos^2(\omega_m t) - \omega_p^2 x_1 = 0. \quad (66)$$

Here, $\omega_m = 2\pi v/\lambda$ is the magnet frequency[4] and $\omega_p = \sqrt{-\eta \varrho_1/\varepsilon_0 \gamma^2}$ is the relativistic plasma frequency of the beam. Since no assumtion has been made on the position of the electron under consideration with respect to the beam, equation (66) is valid for every single beam electron. Understanding x to be the coordinate of a surface electron – i.e. half the beam thickness – this equation is referred to as the beam equation.

6 THE BEAM ENVELOPE

Unfortunately, the nonlinear beam equation, wich can be rewritten as

$$\ddot{x} + \omega_x^2 \left(1 + \cos(2\omega_m t)\right) x - \omega_p^2 x_1 = 0 \quad (67)$$

turns out to be of the Mathieu type. – Obviously, if the braced term was not present, this equation would describe an oscillation in x with the frequency $\omega_x = \omega_0/\sqrt{2}$. – However, assuming that

$$\omega_m \gg \omega_x, \quad (68)$$

which will be discussed later, only the average of the timedependent coefficient should enter this equation, i.e.

$$\ddot{x} + \omega_x^2 \left(x - \frac{\omega_p^2}{\omega_x^2} x_1\right) = 0. \quad (69)$$

Taking into account the initial values $x(t_1) = x_1$ and $\dot{x}(t_1) = \dot{x}_1$ the solution is

$$\frac{x}{x_1} = \frac{\omega_p^2}{\omega_x^2} + \left(1 - \frac{\omega_p^2}{\omega_x^2}\right) \cos \omega_x (t - t_1). \quad (70)$$

Berlin, November 27 1997

[4]The field of the focusing structure appears to the electrons as a timevariant field oscillating with ω_m.

7 REFERENCES

[1] Leon Brillouin, "A Theorem of Larmor and Its Importance for Electrons in Magnetic Fields", Phys. Rev., vol. 67, 260-266, April 1945

[2] H. Busch, "Berechnung der Bahn von Kathodenstrahlen im axialsymmetrischen elektromagnetischen Felde", Annalen der Physik, Band 81, Heft 25, 974-993, 1926

[3] A. S. Gilmour, "Principles of Traveling Wave Tubes" Artech House, Boston/London 1994

[4] J. R. Pierce, "Theory and Design of Electron Beams", D. Van Nostrand Company, Inc., New York, 1954

[5] S. Solyga, "Spreading and Perveance of Round and Sheet Electron Beams", TET-NOTE ????, November 1997