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On the Transverse Dependence of the Shunt Impedance in 2-Dimensional Muffin-Tin Structures

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In the following it is shown, that the shunt impedance of the monopole mode in 2-dimensional muffin-tin structures is generally proportional to $\cosh^2(ky/\beta\gamma)$, where k is the free space wave number, y the transverse offset and β and γ the usual relativistic factors.

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From accelerator theory it is known that for highly relativistic particles there is no transverse dependence of the shunt impedance of slow-wave structures. However, input cavity simulations for a low-voltage klystron have shown that this property does not hold for non-relativistic beams.

In the following it is shown, that the shunt impedance of the monopole mode in 2-dimensional muffin-tin structures is generally proportional to $\cosh^2(ky/\beta\gamma)$, where k is the free space wave number, y the transverse offset and β and γ the usual relativistic factors.

1 Introduction

The interaction between charged particles and resonant or slow-wave structure fields is mainly described by the shunt impedance of the structure. Since it is usually desired that all particles of a beam are equally effected by the fields, the shunt impedance should be constant over the beam thickness.

Unfortunately, for an input cavity structure of a low-voltage sheet beam klystron a constant shunt impedance couldn't be achieved. This fact was somewhat surprising, since this was the case even if the electric field in the gaps had been made flat by varying their width. Further numerical investigations turned out, that the ratio of the impedances at the wall of the beam pipe and at the plane of symmetry is a constant and does neither depend on the the number of cells or their width nor on the phase advance per cell. However, it depends on the aperture of the pipe and the beam voltage.

This simple relation could hardly be accidental and has been investigated analytically.

2 Resonant Structures

Figure 1 shows a two-dimensional multi cell structure. For having it a resonant behaviour,

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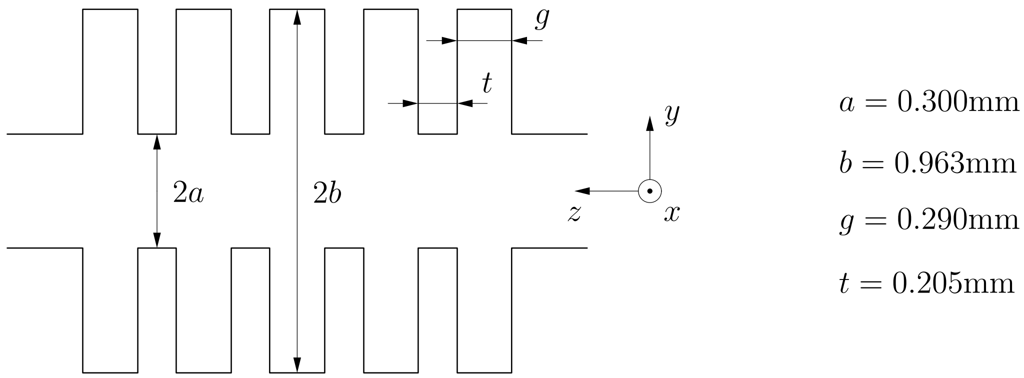

Figure 1: Two-dimensional multi cell π-mode cavity for 91.39GHz

the beam pipe must be operated below cut-off and for $|y| < a$ we expect a steady electric field that vanishes as $z \to \pm\infty$. Therefore, E_z can be represented by its spectrum as

$$E_z(y,z) = \int_{-\infty}^{\infty} A(y,k_z)\mathrm{e}^{-\mathrm{j}k_z z}\mathrm{d}k_z, \quad (1)$$

$$A(y,k_z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_z(y,z)\mathrm{e}^{\mathrm{j}k_z z}\mathrm{d}z. \quad (2)$$

Since E_z must be a solution of the wave equation, setting

$$A(y,k_z) = Y(y)B(k_z) \quad (3)$$

it is required that

$$0 = \int_{-\infty}^{\infty}\left[\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} - (k_z^2 - k^2)Y\right]B(k_z)\mathrm{e}^{-\mathrm{j}k_z z}\mathrm{d}z. \quad (4)$$

For the monopole mode we have $Y(-y) = Y(y)$ and (4) is fulfilled by

$$Y(y) = \cosh\left(\sqrt{k_z^2 - k^2}\,y\right). \quad (5)$$

Thus equations (1) and (2) read

$$E_z(y,z) = \int_{-\infty}^{\infty} B(k_z)\cosh\left(\sqrt{k_z^2-k^2}\,y\right)\mathrm{e}^{-\mathrm{j}k_z z}\mathrm{d}k_z, \quad (6)$$

$$B(k_z)\cosh\left(\sqrt{k_z^2-k^2}\,y\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_z(y,z)\mathrm{e}^{\mathrm{j}k_z z}\mathrm{d}z. \quad (7)$$

Taking the time dependence into account we have

$$E_z(y,z,t) = \mathrm{e}^{\mathrm{j}\omega t}\int_{-\infty}^{\infty} B(k_z)\cosh\left(\sqrt{k_z^2-k^2}\,y\right)\mathrm{e}^{-\mathrm{j}k_z z}\mathrm{d}k_z. \quad (8)$$

Assume a point travelling with a constant velocity u_0 parallel to the z-axis which reaches $z = 0$ at $t = t_0$. Its motion is described by

$$t(z, t_0) = t_0 + \frac{z}{u_0} \tag{9}$$

and the voltage experienced by this point is

$$V(y, t_0) = \int_{-\infty}^{\infty} E_z(y, z, t(z, t_0))\mathrm{d}z, \tag{10}$$

$$= \mathrm{e}^{\mathrm{j}\omega t_0} \int_{-\infty}^{\infty} E(y, z)\mathrm{e}^{\mathrm{j}\omega z/u_0}\mathrm{d}z. \tag{11}$$

But this integral is known, it is just equation (7) for $k_z = \omega/u_0$ and the experienced voltage is therefore

$$V(y, t_0) = 2\pi B(\omega/u_0)\cosh\left(\sqrt{(\omega/u_0)^2 - k^2}y\right)\mathrm{e}^{\mathrm{j}\omega t_0}. \tag{12}$$

This simple result may be somewhat surprising since we got the voltage without actually knowing the electric field. We only have to know its z-related spectrum at $k_z = \omega/u_0$. This may be explained as follows: Equation (8) expresses E_z by an infinite number of waves travelling in z-direction, all having different phase velocities $u_{ph} = \omega/k_z$. But the integrating point is synchronous only with one of these waves, namely that with $u_{ph} = u_0$. Thus, the experienced voltage is proportional to the amplitude $B(\omega/u_0)$ of this synchronous wave. The phase offset between the point and the wave is expressed by $\mathrm{e}^{\mathrm{j}\omega t_0}$.

Making use of the relativistic factors $\beta = u_0/c_0$ and $\gamma^{-2} = 1 - \beta^2$ the wavenumber expressions simplify and the voltage amplitude reads

$$V(y) = 2\pi B(\frac{k}{\beta})\cosh\left(\frac{k}{\beta\gamma}y\right). \tag{13}$$

The constant $B(k/\beta)$ which is still unknown can be determined if the experienced voltage is known for a specific value of y. In the case of a single gap, the voltage at the walls of the beam pipe is usually considered as known which could have been adapted for the more general case considered here. However, since we are interested in the principal dependence only we simply use the voltage at $y = 0$ for normalization

$$V(y) = V(0)\cosh\left(\frac{k}{\beta\gamma}y\right). \tag{14}$$

If the average power dissipated by the walls for a given field is P_d, the shunt impedance of the resonator is defined by $R = V^2/P_d$. Thus, we finally have for the shunt impedance of a two-dimensional resonant structure

$$R(y) = R(0)\cosh^2\left(\frac{k}{\beta\gamma}y\right). \tag{15}$$

3 Periodic Structures

The relations for periodic slow-wave transmission lines are essentially the same, but the expression of the electric field by its spectrum is somewhat harder to derive. Therefore we will make use of Floquet's theorem stating that for a given mode of oscillation, at a given frequency, for all field components the relation

$$f(z+L) = cf(z) \tag{16}$$

holds, where L is the period length of the structure and c a certain complex constant. If the structure is free of losses, c is of magniutude 1 yielding

$$E_z(z+L) = \mathrm{e}^{-\mathrm{j}\Delta\varphi} E_z(z), \tag{17}$$

where $\Delta\varphi$ is the phase advance per cell. Obviously, the function

$$F(z) = \mathrm{e}^{\mathrm{j}\frac{\Delta\varphi}{L}z} E_z(z) \tag{18}$$

is periodic in z with the period length L, since

$$F(z+L) = \mathrm{e}^{\mathrm{j}\frac{\Delta\varphi}{L}z} \underbrace{\mathrm{e}^{\mathrm{j}\Delta\varphi} E_z(z+L)}_{E_z(z)} = F(z) \tag{19}$$

and may be expanded in a Fourier series

$$\mathrm{e}^{\mathrm{j}\frac{\Delta\varphi}{L}z} E_z(y,z,t) = \mathrm{e}^{\mathrm{j}\omega t} \sum_{i=-\infty}^{\infty} A_i(y) \mathrm{e}^{-j\frac{2\pi i}{L}z}. \tag{20}$$

Since E_z must be a solution of the wave equation, for the monopole mode we have

$$A_i(y) = B_i \cosh\left(\sqrt{k_i^2 - k^2}y\right), \tag{21}$$

$$k_i = \frac{\Delta\varphi}{L} + \frac{2\pi i}{L} \tag{22}$$

and the electric field may finally be expressed as

$$E_z(y,z,t) = \mathrm{e}^{\mathrm{j}\omega t} \sum_{i=-\infty}^{\infty} B_i \cosh\left(\sqrt{k_i^2 - k^2}y\right) \mathrm{e}^{-\mathrm{j}k_i z}. \tag{23}$$

If the way l of a point travelling with a constant velocity u_0 is sufficiently long, it experiences the voltage per meter

$$E_{acc}(y,t_0) = \lim_{l\to\infty} \frac{1}{l} \int_{-l/2}^{l/2} E_z(y,z,t(z,t_0))\mathrm{d}z \tag{24}$$

$$= \mathrm{e}^{\mathrm{j}\omega t_0} \sum_{i=-\infty}^{\infty} B_i \cosh\left(\sqrt{k_i^2 - k^2}y\right) \lim_{l\to\infty} \frac{1}{l} \int_{-l/2}^{l/2} \mathrm{e}^{-j(\frac{\omega}{u_0} - k_i)z} \mathrm{d}z. \tag{25}$$

Obviously, if the electric field does not contain a space harmonic with $k_i = \omega/u_0$, this voltage vanishes. Otherwise we have an effective accelarating field of the magnitude

$$E_{acc}(y) = E_{acc}(0) \cosh\left(\frac{k}{\beta\gamma}y\right). \tag{26}$$

With the shunt impedance per meter defined as $r = E_{acc}^2/P_d'$, the final relation is therefore

$$r(y) = r(0) \cosh^2\left(\frac{k}{\beta\gamma}y\right). \tag{27}$$

4 Derivation from Wake Potential

The same results can be derived using the wake potential for the mode under consideration. As described by Vaganian/Henke[1], the wake potential posesses the properties

$$\mathbf{curl}_s \mathbf{W} = \mathbf{0}, \tag{28}$$

$$\text{div}\mathbf{W}_\perp + \frac{1}{\gamma^2}\frac{\partial W_z}{\partial s} = 0. \tag{29}$$

Here, the subscript s indicates differentiation with respect to s instead of z. From (28) follows that $\mathbf{W}$ can be derived from a scalar potential

$$\mathbf{W} = \mathbf{grad}\Psi \tag{30}$$

and (29) defines the equation for this potential

$$\Delta_{\gamma s}\Psi = 0. \tag{31}$$

Assuming again $\partial/\partial x = 0$, for the monopole mode the potential must be of the form

$$\Psi(y,s) = \cosh(k'y)[A'\cos(k'\gamma s) + B'\sin(k'\gamma s)], \tag{32}$$

where k' is a complex constant. The z-component of the wake potential is therefore

$$W_z(y,s) = \cosh(k'y)[A\cos(k'\gamma s) + B\sin(k'\gamma s)]. \tag{33}$$

We now determine the constant k'. Assuming a single mode structure, if the structure is resonant, E_z must be harmonic in time and the longitudinal wake must therefore be harmonic in s. Furthermore, for a lossfree structure, the wake at $s + u_0/f$ must be the same as at s since the the point $s + u_0/f$ experiences the same field as the point s – exactly one rf-cycle later. Thus, the period length of W_z is u_0/f and from $2\pi = k'\gamma u_0/f$ the constant k' is determined to be

$$k' = \frac{k}{\beta\gamma}. \tag{34}$$

But this equation also holds for a synchronous travelling wave, since the wavelength was just u_0/f. Thus, the longitudinal wake is always

$$W_z(y,s) = W_z(y,0)\cosh\left(\frac{k}{\beta\gamma}y\right). \tag{35}$$

Since the voltage experienced by the point moving at s is $-qW_z(y,s)$, we have finally derived equation (14).

5 Practical values

For a 91.3GHz single gap cavity with $a = 0.30$mm, $b = 0.88$mm, $g = 0.79$mm and a particle velocity of $\beta = 0.302$, equation (15) predicts $R(0.2\text{mm}) \approx 3.3R(0)$.

The same geometry has been investigated with the finite difference code GdfidL. For $y \leq 0.25$mm, the relative difference between (15) and the shunt impedances calculated by the code has been less than 0.5%.

[1]Vaganian, S. and Henke, H. The Panofsky-Wenzel Theorem and General Relations for the Wake Potential, TU-Berlin, internal note (TET-NOTE 93/016), December 1993