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On Particle-To-Wave Coupling Coefficients

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In this paper the influence of the particle-to-wave coupling coefficients on the shunt impedance is investigated and simple rules for obtaining correct impedance values are derived.

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Making use of geometry and field symmetries when simulating rf structures with electromagnetics codes like GdfidL or Mafia may result in wrong values for the shunt impedance, since the particle-to-wave coupling coefficients depend on the interaction length in a nonlinear manner.

In this paper the influence of the particle-to-wave coupling coefficients on the shunt impedance is investigated and simple rules for obtaining correct impedance values are derived.

1 Introduction

When simulating rf structures with numeric codes, for saving execution time and memory symmetric properties of the stuctures and the fields are usually taken into consideration. Since the volume to surface ratio is usually kept constant by this procedure, the quality factor is exactly determined by the code. However, the shunt impedance calculation may be influenced this way, since 1) the surface area changes but the interaction length (as seen by the code) is the same or 2) both the surface area and the interaction length changes.

In the first case, the impedance calculated by the code can be corrected afterwards by simply multiplying it with the reduced to original surface ratio. In the second case, the dissipated power can be corrected as in the first case. However, the voltage experienced (by a particle or beam) is a linear function of the reduced to original interaction length ratio only if additional conditions are met, which are subject to the following considerations.

2 Coupling with a single space harmonic

Consider a single space harmonic with a z-component of electric field given by

$$E(z,t) \;=\; \hat{E}\mathrm{e}^{\mathrm{j}(\omega t - kz)} \tag{1}$$

and a particle moving along the z-axis with *constant* velocity u_b,

$$z \;=\; z_0 + u_b(t - t_0). \tag{2}$$

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Then, the electric field experienced by the particle – which is just the field at its position – is $\tilde{E} = E[z, t(z)]$, where $t(z)$ is given by (2). Defining the particle wave number as $k_b = \omega/u_b$, this field reads

$$\tilde{E}(z) = \hat{E}e^{j(\omega t_0 - k_b z_0)}e^{j(k_b - k)z}. \tag{3}$$

Obviously, $\tilde{E}$ is a constant if and only if the particle velocity equals the phase velocity of the wave, $k_b = k$. If additionally the particle passes $z = 0$ at $t = 0$, $z_0 = u_b t_0$, the experienced electric field is simply $\tilde{E} = \hat{E}$.

Furthermore, consider the interaction to occur on a lenght l with the center at $z = z_0$ (this point is passed by the particle at $t = t_0$). Then, the average experienced electric field is given by

$$\bar{E} = \frac{1}{l}\int_{z_0 - l/2}^{z_0 + l/2} \tilde{E}\mathrm{d}z, \tag{4}$$

$$\bar{E} = \frac{\sin{(k_b - k)l/2}}{(k_b - k)l/2}\hat{E}e^{j(\omega t_0 - kz_0)}. \tag{5}$$

The main factor determining the average to maximum field ratio is the particle-to-wave coupling coefficient

$$M = \frac{\sin{(k_b - k)l/2}}{(k_b - k)l/2}, \tag{6}$$

which takes its maximum for $k_b = k$, i.e. if the particle is synchronous with the wave. With (6) and (3), equation (5) can be written as

$$\bar{E} = M\tilde{E}(z_0), \tag{7}$$

stating that the average experienced field is just the product of the coupling coefficient and the field experienced by the particle when passing the center of the interaction region.

3 Coupling with a superposition of space harmonics

Beeing given the electric field

$$E(z, t) = e^{j\omega t}\sum_i \hat{E}_i e^{-jk_i z} \tag{8}$$

and the particle trajectory (2). Since no non-linear operations have been performed when deriving (7), the net average experienced field is given by the superposition of the single averages

$$\bar{E} = \sum_i M_i \tilde{E}_i(z_0) = e^{j\omega t_0}\sum_i M_i \hat{E}_i e^{-jk_i z_0} \tag{9}$$

where the coupling coefficients are given by

$$M_i = \frac{\sin{(k_b - k_i)l/2}}{(k_b - k_i)l/2}. \tag{10}$$

4 Impedances of E_{01p} resonators

As first application of the results from the previous section I will investigate the impedances of rectangular cavity resonators in E_{01p}-mode with respect to their numeric simulation.

By definition, the shunt impedance of such a cavity is given by

$$R = |\bar{E}d|^2/P, \tag{11}$$

where $\bar{E}$ is the average field experienced by a particle, d the cavity and thus the interaction length and P the average power dissipated in the cavity walls.

When the cavity is simulated with numeric codes, symmetries are usually taken into consideration, and thus the surface A of the cavity is reduced to A_c and the interaction length from d to l. Defining the reduction ratios in surface area and interaction length as

$$r_a = A/A_c \tag{12}$$

$$r_l = d/l, \tag{13}$$

the code values subscipted by c, the formula for determining the actual impedance from that predicted by the code reads

$$R = \frac{r_l^2}{r_a}\left|\frac{\bar{E}}{\bar{E}_c}\right|^2 R_c, \tag{14}$$

where it has been assumed, that the reduction in dissipated power equals that in surface area.

Obviously, the main question to answer is that about the actual to code predicted average electric field ratio. If this ratio is not unity, the impedance predicted by the code is usually considered to be uncorrectable, since the code user was forced to evaluate the following formulas by hand.

4.1 E_{010} rectangular resonators

Consider a rectangular cavity resonator of length d excited at E_{010} mode. Its electric field is given by

$$E(z,t) = \hat{E}e^{j\omega t}, \tag{15}$$

which may be considered to be (1) with $k = 0$. Placing the origin of the coordinate system at the center of the cavity, the average experienced voltage is $\bar{E} = M\tilde{E}(0) = M\hat{E}e^{j\omega t_0}$ with the coupling coefficient[1] given by

$$M = \frac{\sin k_b d/2}{k_b d/2}. \tag{16}$$

[1] The coupling coefficient (16) is also called transit time factor, since the reduction in voltage for $k = 0$ may be explained by means of the time it takes to cross the cavity: A particle having a velocity u_b needs a time $\tau = d/u_b = dk_b/\omega$ to cross a gap of the width d. If the frequency of the cavity field was very low, the particle would not consider its change in time, but it does for high frequencies. Thus, the reduction in experienced voltage depends rather on the change in phase $\varphi = \omega\tau = k_b d$ than the transit time. This phase is called transit angle and the reduction can be calculated to be $\sin(\varphi/2)/(\varphi/2)$ which is just (16).

When simulating this cavity with an electromagnetics code, one was attempt to place an electric wall at $z = 0$ and a magnetic wall at $y = 0$, reducing the interaction length to $l = d/2$ and changing the center to $z_0 = d/4$. Since the code field amplitude $\hat{E}_c$ is equal to the actual one $\hat{E}$, the average field calculated by the code was $\bar{E}_c = M_c\tilde{E}(d/4) = M_c\hat{E}\mathrm{e}^{\mathrm{j}\omega t_0}$ with the coupling coefficient

$$M_c = \frac{\sin k_b d/4}{k_b d/4}, \tag{17}$$

and the field ratio was

$$\left|\frac{\bar{E}}{\bar{E}_c}\right| = \left|\frac{M}{M_c}\right|. \tag{18}$$

Obviously, (18) becomes unity only for $k_b = 0$ which meant an infinit particle velocity. Thus, a cavity in E_{010}-mode cannot be correctly simulated when making use of the $z = 0$ symmetry.

4.2 E_{011} rectangular resonators

The field of a rectangular cavity resonator operated at E_{011}-mode is

$$E(z,t) = \hat{E}\sin(\pi z/d)\mathrm{e}^{\mathrm{j}\omega t}, \tag{19}$$

which may be rewritten as

$$E(z,t) = \mathrm{e}^{\mathrm{j}\omega t}(\hat{E}_1\mathrm{e}^{-\mathrm{j}k_1 z} + \hat{E}_2\mathrm{e}^{-\mathrm{j}k_2 z}) \tag{20}$$

with $\hat{E}_1 = -\hat{E}_2 = \mathrm{j}\hat{E}/2$ and $k_1 = -k_2 = \pi/d$, where index 1 refers to the forward and index 2 to the backward wave. Applying the general formulas from section 3, with $z_0 = 0$ the average field becomes

$$\bar{E} = \mathrm{e}^{\mathrm{j}\omega t_0}\mathrm{j}\hat{E}/2(M_1 - M_2), \tag{21}$$

$$M_{1,2} = \frac{\sin{(k_b \mp \pi/d)d/2}}{(k_b \mp \pi/d)d/2}. \tag{22}$$

Applying magnetic walls at $z = 0$ and $y = 0$ for simulation, the interaction length becomes $l = d/2$ and the center moves to $z_0 = d/4$. Thus, the code values are

$$\bar{E}_c = \mathrm{e}^{\mathrm{j}\omega t_0}\mathrm{j}\hat{E}/2(M_{1c}\mathrm{e}^{-\mathrm{j}\pi/4} - M_{2c}\mathrm{e}^{\mathrm{j}\pi/4}), \tag{23}$$

$$M_{1,2c} = \frac{\sin{(k_b \mp \pi/d)d/4}}{(k_b \mp \pi/d)d/4}, \tag{24}$$

yielding for the field ratio

$$\left|\frac{\bar{E}}{\bar{E}_c}\right|^2 = \left|\frac{M_1 - M_2}{M_{1c} - \mathrm{j}M_{2c}}\right|^2. \tag{25}$$

If the particle is synchronous with the forward wave $k_b = \pi/d$, the coefficients for the forward wave become unity and there is no coupling with the backward wave, but the code calculates $M_{2c} = 2/\pi$. Thus, the field ratio for synchronism reads

$$\left|\frac{\bar{E}}{\bar{E}_c}\right|^2_{sync} = \frac{1}{1+4/\pi^2} \approx 0.712. \tag{26}$$

As for the E_{010} resonance, making use of the symmetry with respect to $z = 0$ yields an uncorrectable value for the shunt impedance.

4.3 E_{01p} rectangular resonators

In the general case, the wavenumber is $k = p\pi/d$ and the field may be written as

$$E(z,t) = \hat{E}\mathrm{e}^{\mathrm{j}\omega t}(\mathrm{e}^{-\mathrm{j}kz} + \mathrm{e}^{\mathrm{j}p\pi}\mathrm{e}^{\mathrm{j}kz}) \tag{27}$$

resulting in an average field

$$\bar{E} = \hat{E}\mathrm{e}^{\mathrm{j}\omega t_0}(M_1 + \mathrm{e}^{\mathrm{j}p\pi}M_2), \tag{28}$$

$$M_{1,2} = \frac{\sin(k_b \mp k)d/2}{(k_b \mp k)d/2}. \tag{29}$$

For the simulation, a general interaction length l with a center at z_0 may be considered and the corresponding equations read

$$\bar{E}_c = \hat{E}\mathrm{e}^{\mathrm{j}\omega t_0}(\mathrm{e}^{-\mathrm{j}kz_0}M_{1c} + \mathrm{e}^{\mathrm{j}p\pi}\mathrm{e}^{\mathrm{j}kz_0}M_{2c}), \tag{30}$$

$$M_{1,2c} = \frac{\sin(k_b \mp k)l/2}{(k_b \mp k)l/2}. \tag{31}$$

Thus, the field ratio becomes

$$\left|\frac{\bar{E}}{\bar{E}_c}\right|^2 = \left|\frac{M_1 + \mathrm{e}^{\mathrm{j}p\pi}M_2}{M_{1c} + \mathrm{e}^{\mathrm{j}p\pi}\mathrm{e}^{\mathrm{j}2kz_0}M_{2c}}\right|^2. \tag{32}$$

This equation will be evaluated in two different ways: At first, as for the E_{010}- and E_{011}-resonances the symmetric properties are taken into account and afterwards proper values for l and z_0 are determined making (32) equal to unity.

Making use of the symmetries at $z = 0$ and $y = 0$, we have $l = d/2$ and $z_0 = d/4$ and for synchronous operation $k_b = k$ the ratio reduces to

$$\left|\frac{\bar{E}}{\bar{E}_c}\right|^2_{sync} = \begin{cases} \dfrac{1}{1+4/(p\pi)^2} & \text{for} \quad p = 1,3,\ldots \\ 1 & \text{for} \quad p = 2,4,\ldots \end{cases}. \tag{33}$$

Obviously, the impedance predicted by the code may be simply corrected for even values of p, i.e. if an electric wall is placed at $z = 0$. For odd values of p (magnetic wall at $z = 0$), no simple correction can be applied. However, for large p the R.H.S. of (33) may be replaced by unity – its value takes ≈ 0.96 for $p = 3$ already.

Now I will determine values of l and z_0 for $|\bar{E}| = |\bar{E}_c|$. – Of course, a trivial solution was $l = d$ and $z_0 = 0$, meaning to simulate the full resonator. – Although there may be others, a practical non-trivial solution can be derived from assuming the particle to be synchronous with the forward wave, which is generally the case due to the fact that this is a neccessary condition for achieving maximum shunt impedance. Then, $M_1 = M_{1c} = 1$ and $M_2 = 0$ yielding the neccessary condition $M_{2c} = 0$ or $kl = n\pi$ with n a positive integer number, or more intuitive

$$l = n\lambda/2, \quad n = 1, 2, \ldots. \tag{34}$$

Thus, for having the code calculating the correct average field, the simulated geometry has to be at least half a wavelength long, its length must be a multiple of half the wavelength and the particle velocity must equal the phase velocity of the forward wave.

5 Impedances of periodic structures

The shunt impedance (per unit length) for periodic structures is defined as

$$r = |\bar{E}|^2/P', \tag{35}$$

where $\bar{E}$ is the average electric field experienced by a particle and P' the power per unit length dissipated in the waveguide walls. Since the structure is considered to be infinitively long, which is implied by 'periodic', the interaction length is consequently also considered to be infintively long and the definition of the average experienced field must be modified to

$$\bar{E} = \lim_{l\to\infty} \frac{1}{l} \int_{z_0-l/2}^{z_0+l/2} \tilde{E}(z)\mathrm{d}z. \tag{36}$$

From Floquet's theorem it follows that the electric field of a loss-free periodic structure with period L may be written as

$$E(z,t) = \mathrm{e}^{\mathrm{j}\omega t} \sum_{i=-\infty}^{\infty} \hat{E}_i^f \mathrm{e}^{-\mathrm{j}k_i z} + \hat{E}_i^b \mathrm{e}^{\mathrm{j}k_i z}, \tag{37}$$

where the superscripts f and b stand for the forward and backward wave, respectively and the wave numbers are given by

$$k_i = \frac{\Delta\varphi}{L} + \frac{2\pi i}{L}. \tag{38}$$

For the phase advance per period we have $\Delta\varphi \in [0, \pi]$. As long as $\Delta\varphi$ does not take its limits, all wavenumbers are different and the exponential functions in (37) are linear independent. However, for $\Delta\varphi = 0$ we have $k_i = -k_{-i}$ and for $\Delta\varphi = \pi$ we have $k_i = -k_{-(i+1)}$ so that two functions of (37) become linear dependent, respectively.

Now, a particle having the trajectory (2) experiences the average field

$$\bar{E} = \mathrm{e}^{\mathrm{j}\omega t_0} \sum_{i=-\infty}^{\infty} M_i^f \hat{E}_i^f \mathrm{e}^{-\mathrm{j}k_i z_0} + M_i^b \hat{E}_i^b \mathrm{e}^{\mathrm{j}k_i z_0}, \tag{39}$$

$$M_i^{f,b} = \lim_{l\to\infty} \frac{\sin(k_b \mp k_i)l/2}{(k_b \mp k_i)l/2} = \begin{cases} 1 & \text{for} \quad k_b = \pm k_i \\ 0 & \text{for} \quad k_b \neq \pm k_i \end{cases} \tag{40}$$

Since for $0 < \Delta\varphi < \pi$ all wavenumbers are different, at most one single coupling coefficient can be different from zero, i.e. only one single space harmonic contributes towards the average experienced field. For $\Delta\varphi = 0$ and $\Delta\varphi = \pi$ the forward and backward waves become dependent and two coupling coefficient may be different from zero. However, due to the dependence it holds that only one single space harmonic contributes towards the average experienced field. Therefore, this field reduces to

$$\bar{E} = \mathrm{e}^{\mathrm{j}\omega t_0} \begin{cases} \hat{E}_i^f \mathrm{e}^{-\mathrm{j}k_i z_0} & \text{for} \quad k_b = k_i \\ \hat{E}_i^b \mathrm{e}^{\mathrm{j}k_i z_0} & \text{for} \quad k_b = -k_i \\ 0 & \text{for} \quad |k_b| \neq |k_i| \end{cases} \tag{41}$$

and is different from zero if and only if the electric field contains a space harmonic with a wavenumber equal to that of the particle. Thus, it makes sense to consider only the synchronous case $k_b = k_j$, and the field further reduces to

$$\bar{E}_{sync} = \mathrm{e}^{\mathrm{j}\omega t_0} \hat{E}_j^f \mathrm{e}^{-\mathrm{j}k_j z_0}. \tag{42}$$

For the 0- and π-mode, the R.H.S. of (41) and (42) may be multiplied by 2.

When simulating such a structure with a numeric code, its length must of course be reduced, lets say to l. If its center is z_0, the field calculated be the code is

$$\bar{E}_c = \mathrm{e}^{\mathrm{j}\omega t_0} \sum_{i=-\infty}^{\infty} M_{ic}^f \hat{E}_i^f \mathrm{e}^{-\mathrm{j}k_i z_0} + M_{ic}^b \hat{E}_i^b \mathrm{e}^{\mathrm{j}k_i z_0}, \tag{43}$$

$$M_{ic}^{f,b} = \frac{\sin(k_b \mp k_i)l/2}{(k_b \mp k_i)l/2}. \tag{44}$$

In general, all coupling coefficients are different from zero making the average field difficult to evaluate. Hence, only the synchronous case $k_b = k_j$ is considered here, where the average field is still given by (43), but the coupling coefficients simplify to

$$M_{ic,sync}^f = \frac{\sin \pi(j-i)l/L}{\pi(j-i)l/L}, \tag{45}$$

$$M_{ic,sync}^b = \frac{\sin(\Delta\varphi + \pi(j+i))l/L}{(\Delta\varphi + \pi(j+i))l/L} \tag{46}$$

and the field ratio reads

$$\left|\frac{\bar{E}}{\bar{E}_c}\right|_{sync}^2 = \left|\frac{\hat{E}_j^f}{\sum\limits_{i=-\infty}^{\infty} M_{ic,sync}^f \hat{E}_i^f \mathrm{e}^{-\mathrm{j}k_i z_0} + M_{ic,sync}^b \hat{E}_i^b \mathrm{e}^{\mathrm{j}k_i z_0}}\right|^2. \tag{47}$$

What values should be chosen for l (and z_0) in order to have this ratio equal to unity? Although there may be others, one way to obtain a practical solution is to find a certain lenghth l that makes all coupling coefficients zero (except that for the j-th forward space harmonic, which is equal to unity for all l). Let's start with the forward waves, equation (45). Obviously, the sine becomes zero if the simulated length l is an integer multiple of the period lenghth L,

$$l = nL, \quad n = 1, 2, \ldots. \tag{48}$$

Continuing with equation (46) we immediately see, that coupling with the backward waves cannot be avoided. However, this doesn't neccessarily mean that the values for the shunt impedance become totally wrong, since the ratio (47) still depends on the wave amplitudes $\hat{E}_i^b$.

For travelling wave applications all $\hat{E}_i^b$ vanish and (47) becomes unity already under condition (48). For standing waves we have $|\hat{E}_i^b| = |\hat{E}_i^f|$, where usually the condition $|\hat{E}_0^f| \gg |\hat{E}_i^f|$ applies.